Eigenvalues of non symmetric matrices

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ABSTRACT *Eigen value, or characteristic value, problems are a special case of boundary value problems that are common in engineering problems contexts involving vibrations, elasticity and other oscillating systems. A wide variety of methods are available for solving Eigen value problems. Most are based on two step process. The first step involves transforming the original matrix to simpler one that retains all the original Eigen values. Then iterative methods are used to determine these Eigenvalues. Jacobi method, Given's method and Householder method are used to find Eigen values of symmetric matrix. Aside from symmetric matrix, LR and QR method are used to find Eigen values of non symmetric matrices.This paper discussesthe ccomparison of the speed and accuracy of these methods. The results of this study indicate that the QR algorithm is more successful method for finding the Eigen values of a real nonsymmetric matrix.*

Keywords: *QR method, LR Method, Non-symmetric matrix.*

Introduction: Iterative methods for finding all the Eigen values of nonsymmetric matrices have appeared only recently. The LR algorithm, based on the successive triangular decomposedof a matrix. A sequence of similar matricesis generated whose limit is triangular.A modified procedure for the LR algorithm improves numerical stability. It uses a modified decomposition with interchanges. The QR algorithmmakes use of unitary transformations instead of triangular decomposition. A variation of this algorithm involves a double-shift technique for combining complex conjugate shifts of origin while using Householder's meth od. It was developed for finding complex conjugate Eigen values of real matrices and is known as the double QR procedure.The LR and QR transformations are applicable to the calculation ofgeneral matrices, but a large number of computational operations are required. Therefore, it is desirable to use a matrix of condensed form.Both of these transformations preserve the form of a Heisenberg or almosttriangular matrix.The preliminary reduction of a matrix to Heisenbergform can be accomplished in several ways. However, in this paper we will be concerned with two important methods to solve discussing problems.

Methodology for LR algorithm: The LR algorithm for finding all the Eigen values of an arbitrary matrixis an interpretation of QD scheme. Its basis is the triangular decomposition of a matrix. Thematrixis factorized into the product of a unit left triangular matrix L and a right triangular matrix R such that
LR = A (1.1)
The LR method begins with the original matrix A
A sequence of matrices is then formed such that $A_{k+1}$is derived from $A_k$decomposingit into $L_k$and $R_k$and forming the product of these in reverse order.

$$A_k=L_kR_k \quad , \quad A_{k+1}= R_kL_k \quad (1.2)$$

for k=l,2, …

In this process, a series of similarity transformations are performed on the original matrix A_1,each of which consists ofpremultiplication by a matrix which eliminates the subdiagonal elements and post-multiplication by it inverse.

$$A_k = R_{\{k-1\}}L_{\{k-1\}} = L_{\{k-1\}}^{\{-1\}}A_{\{k-1\}}L_{\{k-1\}} = L_{\{k-1\}}^{\{-1\}} \dots . L_{\{1\}}^{\{-1\}}A_{\{1\}}A_{\{k-1\}}L\_\{1\}\, L_{\{k-1\}} \quad (1.3)$$

Equation (1.3) can be rewritten as

$$L_1L_2 \dots L_{\{k-1\}}A_k = A_1L_1L_2 \dots . L\_\{k-1\} \qquad (1.4)$$

Using$A_k = L_kR_k$and (1.4) it follows that

$$L_1L_2 \dots L_{\{k-1\}}L_kR_k = A_1L_1L_2 \dots . L\_\{k-1\}$$
$$L_1L_2 \dots L_{\{k-1\}}R_{\{k-1\}} = A_1L_1L_2 \dots . L_{\{k-2\}}$$

And so on. If

$$T_k = L_1L_2 \dots L_k \qquad (1.5)$$
$$U_k = R_kR_{\{k-1\}} \dots R_1$$

Where $T_k$is unit left triangular and $U_k$is right triangular, then

$$T_kU_k = L_1L_2 \dots L_{\{k-1\}}(L_kR_k)R_{\{k-1\}} \dots R_1$$
$$= L_1L_2 \dots R_{\{k-1\}}A_k \dots R_1$$
$$= A_1L_1L_2 \dots R_{\{k-1\}}L_{\{k-1\}} \dots R_1 \quad (1.6)$$
$$= A_1^2L_1L_2 \dots R_{\{k-2\}}L_{\{k-2\}} \dots R_1$$
$$= A_1^k$$

The triangular decomposition of $A_1^k$is T_kU_k. All of the matrices $A_k$have the same Eigen values since they are similar. Under certain conditions, $A_k$tends to aright triangular matrix as $k \to \infty$.
The diagonal elements of this righttriangular matrix are the Eigen values of $A_k$appearing in decreasing orderof magnitude from left to right. The problem is that ofdetermining the matrices Land R of (1.1) directly from A without goingthrough any intermediate steps. Therefore(r-1) rows of L and R can be determined by equating the elements in thefirst (r-1) rows of both sides of equation (1.1).The elements are determined in the followingorder: First row of R, second rowof L; second rowof R and so forth .Another order in which they can be determined is t he first row of R,first column of L; second row of R, second column of L, and so forth.
The LR algorithm is not suitable for full matrices because of the highvolume of computations. Thus, it is necessary to reduce the originalmatrix to some condensed form which is invariant with respect to theLRtransformation. One such form is the upper Heisenberg matrix which is analmost triangular matrix with zeros in position (i,j) for i > j + 1.A full matrix may be reduced to Heisenberg form in a stable manner by theuse of similarity transformations.
QR algorithm: The QR transformation usesa decomposition of matrix A into the product of a unitary matrix Q and a right triangular matrix R.
A sequence of matrix defined starting with $A=A_1$, such that

$A_k = Q_kR_k$, $A_{\{k+1\}} = R_kQ_k$(2.1)
For k=1,2,…,n.
A_{k+1} is formed by post multiplying $R_k$by Q_k.This algorithm can be written as a similarity transformation

$$A_k = Q_{\{k-1\}}R_{\{k-1\}} = Q_{\{k-1\}}^{\{H\}}A_{\{k-1\}}Q_{\{k-1\}}$$
$$= Q_{\{k-1\}}^{\{-1\}} \dots . Q_{\{1\}}^{\{-1\}} A_1A_{\{k-1\}}Q_1 \dots . Q_{\{k-1\}} (2.2)$$

Above equation gives

$$Q_1Q_2 \dots Q_{\{k-1\}}A_k = A_1Q_1Q_2 \dots . Q\_\{k-1\} \quad (2.3)$$

Using $A_k = Q_kR_k$ and (2.3),we have

$$Q_1\, Q \dots Q_{\{k-1\}}QR_k = A_1Q_1Q_2 \dots . Q\_\{k-1\}$$

$$Q_1Q_2 \dots Q_{\{k-1\}}R_{\{k-1\}} = A_1Q_1Q_2 \dots . Q_{\{k-2\}} \ (2.4)$$

And so on.

If

$$P_k = Q_1Q_2 \dots Q_k \quad (2.5)$$
$$S_k = R_kR_{\{k-1\}} \dots R_1$$

Where $T_k$is unitary and $S_k$is right triangular, then

$$P_kS_k = Q_1Q_2 \dots Q_{\{k-1\}}(Q_kR_k)R_{\{k-1\}} \dots R_1$$
$$= Q_1Q_2 \dots R_{\{k-1\}}A_k \dots R_1$$
$$= A_1Q_1Q_2 \dots R_{\{k-1\}}Q \dots R_1 \quad (2.6)$$
$$= A_1^2Q_1Q_2 \dots R_{\{k-2\}}Q_{\{k-2\}} \dots R_1$$
$$= A_1^k$$

Francis proved that for any matrix A there exists a unitarymatrix Q such that A= QR where R is a right triangular matrix which has real non-negative, diagonal elements. Moreover, the Q is unique if A isnon-singular. Thus the unitary-triangular decomposition of any square matrix exists,and, if the matrix is non-singular, the decomposition isunique.

Francis proved that if A is a non-singular matrix with allEigen values of distinct modulus then, if $k \to \infty$,the elements of $A_k$belowthe diagonal tend to zero, and the elements on the diagonal tend to theEigen values of A.The factorization of a matrix $A_k$into $Q_k$and $R_k$involves the use ofelementary unitary transformations

Conclusion: The results of this study indicate that the QR algorithm is a more successful methodthan the LR algorithm for finding the Eigen values ofreal non symmetric matrices. The convergence properties of the modified LR algorithm appliedto real matrices; are not satisfactory except when the Eigen values of thematrix are all real. Another restriction is that for convergence to takeplace any matrix with equal Eigen values must have linear divisors. Also, few matrices which are ill-conditioned with respect to theirEigen values were found to become progressively more ill-conditioned at everyiteration points. The total number of iterations in the convergence to the Eigen valuesmust be taken into consideration when comparing the times required bytheLR and QR algorithms. As the dimension of the testmatrices used was increased, the number of iterations required for convergenceby the LR increased at a faster rate than for the QR. When the total number of iterations is the same for both algorithms,the double QR requires more time. This is again due to the number ofmultiplications involved in one iteration point.
The LR transformation is much easier to apply, but becauseof its possible numerical instability and other restrictions, its valueseems limited. Therefore, the QR algorithm was found to be themore successful method for dealing with the real non symmetric Eigen valueproblem.

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