Solving Differential Equations by using Laplace Transforms

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Received: May 28, 2018 Accepted: July 09, 2018

ABSTRACT *The concept of Laplace transforms plays a vital role in wide fields of science and technology Such as electric and communication engineering, Quantum Physics ,solution of partial differential operation ,ect. this paper provides the reader with a solid foundation in the fundamentals of Laplace transforms & gain on understanding of same of the very important and basic applications of this transforms. Laplace transform is broadly used to solve linear differential equations, particularly initial value problems. Laplace transform reduces a ordinary linear differential equation to an algebraic equation.*

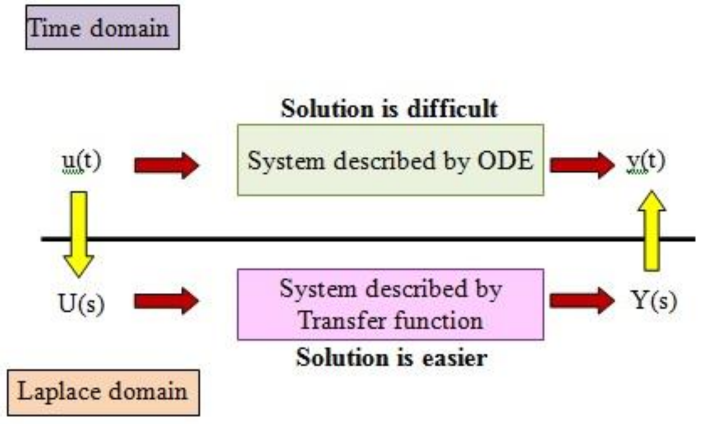
Keywords: *Laplace transforms, Initial value problems, Linear differential equations*

INTRODUCTION :

Laplace Transformation deals in mathematics with the conversion of one function to another function that may be not in the same domain. Laplace transforms is powerful Transformation tool, which literally transforms the original Differential Equation into an elementary algebraic expression, into the Solution of original problem. The transform is named after the mathematician and renowned astronomer Pierre Simon Laplace Who lived in France.

The Transformation method is a logically for solving linear differential equations with initial conditions. It is generally used electrical circuit and systems problems. By using the Laplace transform operator we can solve (first- and second-order) differential equations with constant coefficients. The differential equations must be Initial value problems with the initial conditions specified at x = 0.

The method is easy to explain. Given a value problems apply the Laplace transform operator to both sides of the differential equation. This will Transformation the differential equation into an *algebraic* equation is the Laplace transform of the desired solution. Once you solve this algebraic equation, take the inverse Laplace transform of both sides. The result is the solution to the original Initial value problems .Before this process is undertaken, it is necessary to see what the Laplace transform operator does to y' and y''. Integration by parts yields.



Formulation:

The Laplace Transform can be used to solve differential equations using a four step process.

1) Given the Laplace Transform of the differential equation using the derivative property as necessary.
2) Applying initial conditions into the resulting equation.
3) Solve for the output variable.
4) Then we get the result from Laplace Transform tables.
5) Then we apply Inverse Laplace Transform.

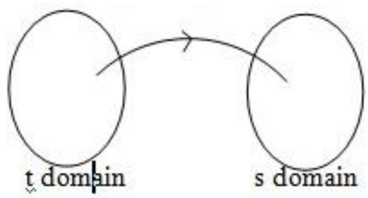
Laplace Transform Definition:

The Laplace Transform Let f be a function defined for t ≥ 0, and. The Laplace Transform of f is defined as

$$L\{F(t)\}=f(p)=\int_0^\infty e^{-pt}\ F(t)\ dt$$

Here 'p' is real or complex number.

L{F(t)}=f(p)



Elementary properties of Laplace Transform:

1. **Linearity Property:** If L{F(t)}=f(p) and L{G(t)}=g(p) then
   $L\{C_1 F(t)+ C_2 G(t)\} = C_1 L\{F(t)\}+ C_2 L\{G(t)\} = C_1 f(p)+ C_2 g(p)$

Where C1 and C_2 are Constants.

2. **First shifting theorem:** If L{F(t)} = f(p) then $L\{e^{at} F(t)\} = f(p-a)$, p-a > a
3. **Second shifting theorem:** If L{F(t)} = f(p) then and $G(t)=\begin{cases} f(t-a), & t < a \\ 0\quad, & t > a \end{cases}$ then
   $L\{G(t)\} = e^{-ap}F(p)$
4. **Laplace Transformation of derivatives;** If L{F(t)} = f(p),then
   $L\{F^{|}(t)\} = pf(p)-F(0)$
5. **Laplace Transformation of integrals;** If L{F(t)} = f(p) then
   $L\{\int_0^\infty\ F(u)\ du\} = \frac{f(p)}{p}$
6. **Multiplication by t^n:** If L{F(t)} = f(p),then $L\{t^nF(t)\}= (-1)^n f^n (p)$
7. **Division by t**: If L{F(t)} = f(p),then $\frac{F(t)}{p} = \int_0^\infty\ f(u)\ du$

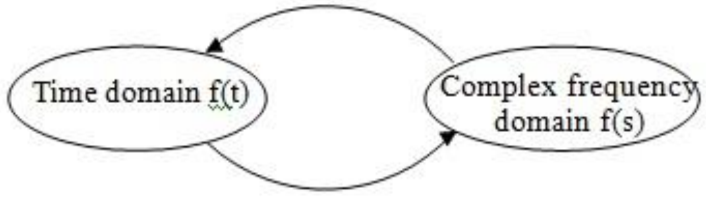
Laplace transform of some other Elementary Functions

F(t)	f(s)=L{F(t)}
1	
t^n	$\frac{n!}{S^{n+1}}$
e^{at}	$\frac{1}{p-a}$
sin (at)	$\frac{a}{p^2+a^2}$
cos (at)	$\frac{p}{p^2+a^2}$
sinh(at)	$\frac{a}{p^2-a^2}$
cosh(at)	$\frac{p}{p^2-a^2}$

Table-1

Inverse Laplace Transform Definition: If f(p) is the Laplace Transform of a function F(t) ,then F(t) is called the inverse Laplace Transform of f(p) and is denoted by $L^{-1}\{f(p)\}$

i.e., $L^{-1}\{f(p)\} = F(t)$



Elementary properties of Inverse Laplace Transform:

1. **Inverse Laplace Transform Linearity Property:** If $F(t)= L^{-1}\{f(p)\}$ and $G(t)= L^{-1}\{g(p)\}$ then $L^{-1}\{ C_1 f(p)+ C_2 g(p) \} = C_1 L^{-1} \{ f(p)\}+ \mathbf{C_2} L^{-1}\{ g(p)\}$

Where C_1 and C_2 are Constants.

2. **Inverse Laplace Transform first shifting theorem:** If $L^{-1}\{f(p)\} = F(t)$ then $L^{-1}\{f(p-a)\} = e^{at} F(t)$ p-a > a
3. **Inverse Laplace Transform Laplace Transformation of derivatives;** If $L^{-1}\{f(p)\} = F(t)$ then $L^{-1}\{f^n(p)\} = (-1)^n t^n F(t)$
4. **Inverse Laplace Transform Laplace Transformation of integrals;** If $L^{-1}\{f(p)\} = F(t)$ then $L^{-1}\{\int_0^\infty f(p)\, dp\} = \frac{F(t)}{t}$
5. **Inverse Laplace Transform Multiplication by p** : If $\mathbf{L}^{-1}\{f(p)\} = F(t)$ and $f(0)=0$ then $L^{-1} \{p f(p) \}=F^{|}(t)$
6. **Inverse Laplace Transform Division by p**: If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\{\frac{f(p)}{p}\}= \int_0^\infty F(u)\, du$
7. **Inverse Laplace Transform of product of two Functions:** If $L\{F(t)\}=f(p)$ and $L\{G(t)\}=g(p)$ Then $L\{F(t) * G(t)\} = f(p)g(p)$ or $\{F(t) * G(t)\} = L^{-1} f(p)g(p)$ Which is known as Convolution of F and G.

Applications of Laplace Transform:

The Laplace transform technique is applicable in many fields of science and technology such as:

- Control Engineering
- Communication
- Signal Analysis and Design
- System Analysis
- Solving Differential Equations

Solving Ordinary Differential Equations:

Problem:

$Y^{||}+aY^{|}+bY=G(t)$ Subject to the initial condition $Y(0)=A$,$Y^{|}(0)=B$ Where a,b,A,B are Constants.

Solution:

1) Apply the operator L to both sides of the differential equation; then use linearity, the initial conditions, and Table 1 to solve for $L[y]$
2) Next we apply inverse Laplace transform Operator
3) More simply, we obtain the desired solution Y.

Ex: Solve $ty^{||} + y^{|} + ty = 0$ gives that Y(0)=1.

Sol: Taking the Laplace transform of the given equation, we get

$$L\{ty^{||} + y^{|} + ty\}=0$$

$$L\{ty^{||}\} + L\{y^{|}\} + L\{ty\}=0$$

By using Multiplication by t^n then

$$(-1)\frac{d}{dp}L\{y^{||}\} + \{pL\{y\}-y(0)\} + (-1)\frac{d}{dp}L\{y\} = 0$$

By using Laplace Transformation of derivatives,

$$(-1)\frac{d}{dp} [p^2L\{y\} - py(0) - y^{|}(0)]+ \{pL\{y\} - y(0)\} + (-1)\frac{d}{dp}L\{y\} = 0$$

using the given condition ,it reduces to

$$(-1)\frac{d}{dp} [p^2L\{y\} - p .1 - a]+ \{pL\{y\} - 1\} + (-1)\frac{d}{dp}L\{y\} = 0$$

(Since $Y^{|}(0)$ is not given ,therefore we take $Y^{|}(0)=a$)

i.e., $-\frac{d}{dp} [p^2 Z - p .1 - a]+ \{p Z - 1\} - \frac{d}{dp} Z = 0$ Where

$$- p^2 \frac{dZ}{dp} - 2pZ + 1 + pZ - \frac{dZ}{dp} = 0$$

$$\frac{dZ}{dp}(p^2+1) +pZ$$

By using variables separable

$$\frac{dZ}{dp}(p^2+1) = - pZ$$

$$\frac{1}{Z} dZ = - \frac{p}{p2+1} dp$$

Integrating on both sides

$$\int \frac{1}{Z}\, dZ + \int \frac{p}{p2+1}\, dp$$

$$\log Z + \frac{1}{2}\int \frac{2p}{p2+1}\, dp$$

$$\log Z + \log(p^2+1) = \log c$$

$$Z(p^2+1) = c$$

$$Z = \frac{c}{(p2+1)}$$

$$L\{y\} = \frac{c}{(p2+1)}$$

$$y = L^{-1}\{\frac{c}{(p2+1)}\} = c.J_0(t)$$

we have $Y(0) = 1$

$$1 = c \,.\, J_0(t) = c.1$$

Since $J_0(0)=1$

$$\Rightarrow c=1$$

Hence the required solution is $y = J_0(t)$

Conclusion:

The Laplace transform is powerful tool using in different areas of mathematics, physics and engineering. With the ease of application of Laplace transforms in many applications, much research software has made it possible to simulate the Laplace transformable equations directly which has made a good advancement in the research field. I also how to solve an initial boundary value problems by using Laplace transform. The Laplace transformation method has been successfully applied to find an exact solution of fractional ordinary differential equations, with constant and variable coefficients.

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