The Independent Pair Resolving Sets in Graphs

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ABSTRACT *In this paper, we define the independent pair resolving sets in graphs. We study the existence of independent pair resolving sets in graphs and characterized all nontrivial connected graphs G of order n with ipr(G)=1,n-1,n-2, and we have presented some results.*

Keywords: *Independent set, Pair Resolving set, Independent PR-set, Independent PR-number.*

1. Introduction

The distance between u and v in G, denoted by $d(u,v)$, is the lenght of the shortest path u to v in G. Let $W=\{w_1,w_2,\ldots\ldots\ldots w_k\}$ be an ordered subset of $V(G)$. For a vertex $v\in V(G)$, a representation of v with respect to W is k-tuple $r(v/W)=(d(v,w_1),d(v,w_2),\ldots\ldots..d(v,w_k))$. Metric dimension was initially introduced in 1970s, by Harary and Melter [3], and independently by Slater [8]. Slater introduced these ideas and used locating set for what we have called resolving set. He referred to the cardinality of a minimum resolving set in a graph G as its location number of Slater described the usefulness of these ideas when working with U. S. sonar and coast guard Loran (Long range aids to navigation) stations. Independently, Harary and Melter [3], investigated these concepts as well, but used metric dimension rather than location number, the terminology that we have adopted. In general, a connected graph G can have many resolving sets. In this paper, we consider those resolving sets whose vertices are located "close" or "equal" to one another. The independent sets of maximum cardinality are called maximum independent sets and these are the independent sets that have received the most attention. A matching in a graph is a set of non-adjacent edges. A maximum matching is a matching of largest cardinality. The matching number of a graph $\mu(G)$ is the cardinality of a maximum matching. A vertex cover in a graph is a set of vertices that covers all the edges of a graph. The vertex cover number $\tau(G)$ is the cardinality of a minimum vertex cover.

A set of vertices of a graph is a dominating set if every vertex in the graph is dominated by at least one vertex from the set. The domination number $\gamma(G)$ is the minimum cardinality among the dominating sets of a graph G. The number of vertices in a maximum independent set in a graph G is the independence number (or vertex independence number) of G and is denoted by $\alpha(G)$. For a vertex v in a graph G, let $N(v)$ be the set of vertices adjacent to v and let $N[v]=N(v)\cup\{v\}$. A major vertex v of G is an exterior major vertex of G if it has positive terminal degree. Let $\sigma(G)$ denote the sum of the terminal degrees of the major vertices of G and let $ex(G)$ denote the number of exterior major vertices of G. In fact, $\sigma(G)$ is the number of end-vertices of G. A connected graph with exactly one cycle is called a unicyclic graph. The set W is called a *pair resolving set* for G if $u\in V(G)$ then $r(u/W)=r(v/W)$ for at most one v such that $v\in V(G)$. The minimum cardinality of a pair resolving set of G is called the metric dimension of G, denoted by $dim_{pr}(G)$. In this paper, we define the independent PR-set and independent PR-number of a graph with some additional property and we study some related theorems.

2. Independent PR-sets

Definition 2.1 An independent pair resolving set W in a connected graph G is both pair resolving and independent. The cardinality of a minimum independent pair resolving set (or simply an independent PR-set) in a graph G is the independent PR-number $ipr(G)$.

Remark 2.2 Let G be a connected graph of order n containing an independent PR-set. Since every independent PR-set of G is a pair resolving set, it follows that

$$1\le dim_{pr}(G)\le ipr(G)\le\alpha(G)\le n-1$$

Example 2.3 Consider the graph G in Figure.1, The set $S=\{v_1,v_3,v_5\}$, $W=\{v_7,v_8\}$, $W'=\{v_1,v_4\}$. The codes of the vertices of G with respect to W are $r(v_1/W)=(1,1)$, $r(v_2/W)=(2,1)$, $r(v_3/W)=(2,1)$, $r(v_4/W)=(1,1)$, $r(v_5/W)=(1,2)$, $r(v_6/W)=(1,2)$, $r(v_7/W)=(0,1)$, $r(v_8/W)=(0,1)$. The codes of the vertices of G with respect to W' are $r(v_1/W')=(0,2)$, $r(v_2/W')=(1,2)$, $r(v_3/W')=(2,1)$, $r(v_4/W')=(2,0)$, $r(v_5/W')=(2,1)$, $r(v_6/W')=(1,2)$, $r(v_7/W')=(1,1)$, $r(v_8/W')=(1,1)$. We can show that $\alpha(G)=3, pr(G)=2, ipr(G)=2$.

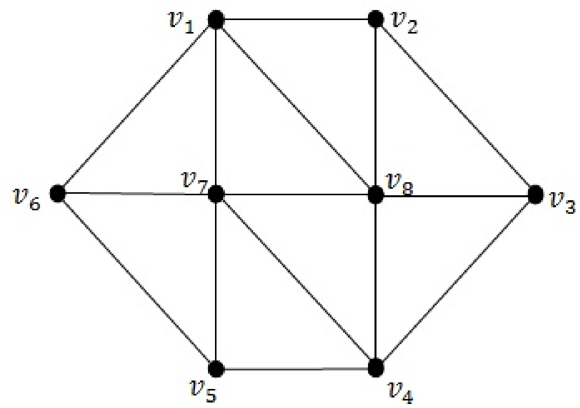

Figure: 1

Theorem 2.4. Let G be a connected graph of order $n \geq 6$ for which $ipr(G)$ is defined. If W is an independent PR-set of G, then $deg(w) \leq n-4$ for every $w \in W$.

Proof. Assume the contrary that there exists $u \in W$ such that $deg(u) \geq n-2$. Since W is independent PR-set, $|W| \leq 2$. On the other hand, since $deg(u) \geq n-2 \geq 4$, it follows that $G \neq P_n$. Since P_n is the only connected graph of odd order n with dimension, it follows that $|W| = 1$. Suppose that $|W| = 2$ and let $W = \{u, v\}$. For each $x \in V(G) - W = N(u)$, the code $pr(x/W) = (d(u,x), d(v,x)) = (1, d(v,x))$. Since $d(v,x)$ is one of $d(u,v), d(u,v)+1$, and $d(u,v)-1$, there are at most three distinct codes for the vertices in $V(G) - W$. However, $|V(G) - W| = |N(u)| = n-2 \geq 4$, a contradiction.

Theorem 2.5. Let G be a connected graph of order $n \geq 6$.
(a) If G contains two nonadjacent vertices of degree $n-2$, then $ipr(G)$ is not defined.
(b) If G contains two vertices of degree $n-1$, then $ipr(G)$ is not defined.

Proof. (a) Assume the contrary that $ipr(G)$ is defined, and let W be an independent PR-set of G. First, suppose that G contains two nonadjacent vertices x and y of degree $n-2$. Then x and y belong to W. Then W contains at least one of x and y, which contradicts theorem 2.4. Thus (a) holds.
(b) Next, suppose that G contains two vertices x and y of degree $n-1$. Then x and y belong to the same distance similar equivalence class in G. Necessarily, W contains exactly one of x and y, which again contradicts theorem. Thus (b) holds.

Theorem 2.6. Let G be a connected graph of order $n \geq 4$. Suppose that G contains two distinct pair resolving sets W_1 and W_2 of cardinality at least 2. If some vertex of W_1 is adjacent to a vertex of W_2, then $ipr(G)$ is not defined.

Proof. Suppose that $u_1u_2 \in E(G)$, where $u_1 \in W_1$ and $u_2 \in W_2$. Since W_1 and W_2 are pair resolving sets, u_1 is adjacent to every vertex of W_2 and so every vertex of W_1 is adjacent to every vertex of W_2. Since every pair resolving set of G must contain at least one vertex from each of W_1 and W_2. This implies, however, that no pair resolving set of G is independent and so $ipr(G)$ is not defined.

Theorem 2.7 If G is a graph; then $dim_{pr}(G) \geq \sigma(G) - ex(G)$.

Proof. Let W be any pair resolving set and let v be an exterior major vertex of G. Let $k = ter(v)$ and let $u_1, u_2, \ldots, u_k$ denote the terminal vertices of v. Thus the branch of G at v containing $u_i, (1 \leq i \leq k)$ is a $v - u_i$ path Q_i. We claim that W contains more than one vertex from each of the paths $Q_i - v, (1 \leq i \leq k)$ with more than one exception. Suppose, to the contrary, that two of these paths contain only one vertex of W, say $Q_1 - v$ and $Q_2 - v$. Let u_1' and u_2' be the vertices adjacent to v on Q_1 and Q_2, respectively. Since $Q_1 - v$ and $Q_2 - v$ contains one vertex of W, it follows that $r(u_1'/W) \neq r(u_2'/W)$; contradicting the fact that W is a pair resolving set. Thus, as claimed, W contains more than one vertex from each of the paths $Q_i - v, (1 \leq i \leq k)$ with more than one exception. Consequently, $dim_{pr}(G) \geq \sigma(T) - ex(T)$.

Theorem 2.8 For a cycle C_n of even order $n \geq 4$, every independent PR-set of C_n is a basis of C_n.

Proof. Since $dim_{pr}(C_n) = ipr(C_n) = 2$ for $n \geq 4$ and every independent PR-set of C_n consists of two non-adjacent vertices. Then every independent PR-set is also a basis.

Theorem 2.9 Let G be a connected graph of order $n \geq 4$.
(a) If $G = C_n$ for even order $n \geq 4$, then $ipr(G)$ =2.
(b) If G is a tree, then $ipr(G) = 1$ if G is a path of odd order and $ipr(G) \geq \sigma(T) - ex(T)$ otherwise.
Proof. a) If G is a cycle C_n of even order $n \geq 4$. Then $dim_{pr} = 2$ and its pair resolving set consists of only two non-adjacent vertices. Therefore both are independent and $ipr(G) = 2$.
b) If G is a path of odd order, then the midpoint of G is a basis element and it is independent. By Theorem 2.8, $ipr(G) = 1$. Otherwise, by Theorem 2.7, $ipr(G) \geq \sigma(T) - ex(T)$.

Theorem 2.10 Let G be a nontrivial connected graph of order n for which $ipr(G)$ exists. Then
(a) $ipr(G) = 1$ if and only if $G = P_n$ of odd order n,
(b) $ipr(G) = n - 2$ if and only if $n = 4$ and $G = K_4$.
Proof. (a) It is an immediate consequence of the fact that P_n is the only connected graph of odd order n with dimension 1.
(b) It is clear that $ipr(K_4) = 2$. For the converse, let G be a connected graph of order n with $ipr(G) = n - 2$. Then $\alpha(G) = n - 1$ and if $n \geq 4$, then $ir(K_n) = n - 2$. Therefore, $n = 4$ and $G = K_4$.

Theorem 2.11 If G is a connected graph with $ipr(G) = \alpha(G)$, then $n - 2\mu(G) \leq ipr(G) \leq n - \mu(G)$.
Proof. Suppose G is a connected graph and let M be a maximum matching. Consider the set of vertices, $W' = V(G) - V(M)$. Then W' is an independent PR-set, since if any two vertices of W' were adjacent then we could add the edge between them to M. So $ipr(G) \geq |W'|$ and $n = |W'| + |V(M)|$, so $|W'| = n - |V(M)| = n - 2\mu(G)$. Then $ipr(G) \geq |W'| = n - 2\mu(G)$. Hence $ipr(G) \geq n - 2\mu(G)$. Also, given a set of independent edges M separating vertices $V(M)$, the largest possible independent set of vertices in $V(M)$ has at most $|M|$ vertices. Then any independent set has at most $|W'| + |M| = |W'| + \mu(G)$ vertices. So $ipr(G) \leq |W'| + \mu(G)$. Then $ipr(G) \leq |W'| + \mu(G) = n - 2\mu(G) + \mu(G) = n - \mu(G)$. Thus $ipr(G) \leq n - \mu(G)$ and therefore $n - 2\mu(G) \leq ipr(G) \leq n - \mu(G)$.

Theorem 2.12 If G is any graph with $ipr(G) = \alpha(G)$, then $ipr(G) \geq n/[1 + \Delta(G)]$.
Proof. Suppose W is an independent PR-set with $|W| = \beta(G)$. Then $V(G) = W \cup \bigcup_{v \in W} N(V)$, since if some vertex $v \in V(G)$ did not belongs to $W \cup \bigcup_{v \in W} N(V)$ then we could add v to our independent PR-set W, which is already maximum. This implies, since $|\bigcup_{v \in W} N(V)| \leq \sum_{v \in W} |N(v)|$ and $|N(v)| \leq \Delta(G)$, for any vertex $v \in V(G)$, that $n = |W| + |\bigcup_{v \in W} N(V)| \leq |W| + |W| \cdot \Delta(G) = ipr(G) + ipr(G) \cdot \Delta(G)$. Then $n \leq ipr(G) + ipr(G) \cdot \Delta(G) = ipr(G)[1 + \cdot \Delta(G)]$. Therefore we get $n/[1 + \Delta(G)] \leq ipr(G)$.

Theorem 2.13 If G has no isolated vertices with $ipr(G) = \alpha(G)$, then $ipr(G) + \tau(G) = n$.
Proof. Suppose G has no isolated vertices and let W be a independent PR-set and let C be a minimum vertex cover. Then $V(G) - W$ is a vertex cover of G since if any vertex was not covered then we could add it to our independent set W. So $\tau(G) \leq n - ipr(G)$ or equivalently, $ipr(G) + \tau(G) \leq n$. Also, $V(G) - C$ is an independent set of vertices, since if two vertices were adjacent then C would not cover the edge between them. So $ipr(G) \geq n - \tau(G)$ or equivalently, $ipr(G) + \tau(G) \geq n$. Thus $ipr(G) + \tau(G) \leq n$
$\leq ipr(G) + \tau(G)$ and therefore, $ipr(G) + \tau(G) = n$.

Theorem 2.14 If G is any graph with $ipr(G) = \alpha(G)$, then $ipr(G)\chi(G) \geq n$.
Proof. Suppose $\chi(G) = k$ and let $V_1, V_2, \ldots, V_k$ be subsets of the vertex set such that V_i is assigned color i under a proper k-coloring of G. Then each Vi is an independent set of vertices, so $ipr(G) \geq M = max\ \{|V_1|, |V_2|, \ldots, |V_k|\}$. Also since $V_1, V_2, \ldots, V_k$ is a partition $V(G)$, we have $M \cdot k \geq n$. Then $ipr(G)\chi(G) \geq M \cdot k \geq n$. Therefore $ipr(G)\chi(G) \geq n$.

Theorem 2.15 If G is any graph with $ipr(G) = \alpha(G)$, then
a) $ipr(G) \geq \gamma(G)$.
b) $ipr(G) \leq n - \delta(G)$.

Proof. a)Suppose W is an independent PR-set of vertices G. Since W is a maximum independent set, then W is a dominating set. Since if any vertex was not dominated by W then that vertex does not belong to $W \cup N(W)$, so we could add it to our independent set W. Therefore $\gamma(G) \leq ipr(G)$.
b) Let $v \in W$. Then $W \cap N(v) = \phi$, so $W \subseteq V(G) \backslash N(v)$, which implies $|W| \leq |V(G) \backslash N(v)| = n - |N(v)| \leq n - \delta(G)$. Thus $ipr(G) = |W| \leq n - \delta(G)$ and therefore, $ipr(G) \leq n - \delta(G)$.

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