Simple Proofs of Bolzano-Weierstass Theorem and Completeness of $R^m$

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ABSTRACT

Using the result ‘every sequence of real numbers has a monotone subsequence’ we prove: If a bounded sequence has unequal lim sup and lim inf, then it has at least two monotone (convergent) subsequences whose range sets are disjoint; (B. W. theorem) Every bounded sequence has a convergent subsequence, and $R$ is a complete space under usual metric on $R$. From these we obtain easy proofs of (B. W. theorem): Every bounded sequence in an Euclidean space has a convergent subsequence, and every Euclidean space is complete.

Keywords:

1. Sequences of real numbers:

A sequence is a function whose domain is $N = \{1, 2, 3, \ldots \}$. If $s: N \rightarrow K (= R$ or $C)$ then $s$ is a sequence in $K$ and it is denoted by $\{s_n\}_{n=1}^{\infty}$ or $\{s_n\}$ or $s_n > 0$ or $s_n \leq 0$. If $s_n \in R \forall n$ then $\{s_n\}$ is a sequence of real numbers etc. The range of a sequence is $\{s_n\}_1^n \rightarrow \infty$. A sequence is convergent if its limit exists and we write $\lim_{n \rightarrow \infty} s_n = l$ or $\lim_{n \rightarrow \infty} s_n = l$ or $s_n \rightarrow l$ as $n \rightarrow \infty$, and we say that $\{s_n\}$ is a convergent sequence (converges to $l$).

A sequence is convergent if and only if it is bounded, i.e. $\{s_n\}$ is a convergent sequence if and only if $\{s_n\}$ is bounded. LUB Axiom: If $A$ is a nonempty subset of $R$ which is bounded above, then it has a least upper bound in $R$, i.e. lub $A \in R$.

Using lub axiom we have; GLB Property: If $B$ is a nonempty subset of $R$ which is bounded below, it has greatest lower bound in $R$, i.e. glb $A \in R$.

In this section we consider sequences in $R$. A sequence $\{s_n\}$ is bounded means its range is bounded, i.e. there is a bound $M > 0$ such that $|s_n| \leq M$ for all $n \in N$. A sequence $\{s_n\}$ is said to be increasing if $s_n \leq s_{n+1}$ for all $n \in N$. A sequence $\{s_n\}$ is said to be decreasing if $s_n \geq s_{n+1}$ for all $n \in N$.

1.2 Theorem: A sequence $\{s_n\}$ is convergent if and only if it is bounded.

Proof: Let $\{s_n\}$ be a convergent sequence and its limit is a number $c$. Then for $\epsilon > 0$, there is an $N \in N$ such that $|s_n - c| < \epsilon$ for $n \geq N$. Hence the sequence $\{s_n\}$ is bounded. If a sequence is bounded and has a convergent subsequence, then it is convergent.

1.3 Algebra of limits and some theorems:

Let $c = (c, c, c, \ldots)$ be a constant sequence and $\lim_{n \rightarrow \infty} s_n = l$, $\lim_{n \rightarrow \infty} t_n = m$. Then

(i) $\lim_{n \rightarrow \infty} c = c$. (ii) $\lim_{n \rightarrow \infty}(s_n + t_n) = l + m$. (iii) $\lim_{n \rightarrow \infty} cs_n = cl$. (iv) $\lim_{n \rightarrow \infty}s_n^k = l^k$ for $k \in N$. (v) $\lim_{n \rightarrow \infty}s_n \cdot t_n = l \cdot m$.

(vi) $\lim_{n \rightarrow \infty}\frac{s_n}{m} = 1$. provided $m \neq 0$. (vii) $\lim_{n \rightarrow \infty}\frac{s_n}{t_n} = l$. provided $m \neq 0$. (viii) $\lim_{n \rightarrow \infty}s_n^{p}$ for any fixed $p \in N$.

(ix) $\lim_{n \rightarrow \infty}|s_n| = |l|$. (x) $\lim_{n \rightarrow \infty}s_n = 0$ iff $\lim_{n \rightarrow \infty}|s_n| = 0$. (xi) If $s_n \geq 0 \forall n \in N$ (or $s_n \geq 0 \forall n \in N$ for some fixed $n_0 \in N$) $\Rightarrow l \geq 0$.

(xii) $\lim_{n \rightarrow \infty}t_n = \lim_{n \rightarrow \infty}n_0$. (xiii) $\lim_{n \rightarrow \infty}n_0$ in $N$ is fixed $\Rightarrow l = m$.

(xiv) $\lim_{n \rightarrow \infty}a^n = 1$, for any fixed real number $a > 0$.

Sandwich Theorem: If $s_n \leq t_n \leq u_n \forall n \in N$ and $\lim_{n \rightarrow \infty}s_n = l$, $\lim_{n \rightarrow \infty}u_n = l$ then $\lim_{n \rightarrow \infty}t_n = l$.

Monotonic sequences: Let $\{s_n\}$ be a sequence of real numbers. It is called

(i) a monotonically increasing (m. i.) if $s_n \leq s_{n+1} \forall n \in N$. (ii) strictly increasing if $s_n < s_{n+1} \forall n \in N$.

(iii) monotonically decreasing (m. d.) if $s_n \geq s_{n+1} \forall n \in N$. (iv) strictly decreasing if $s_n > s_{n+1} \forall n \in N$.
Note also that increasing = non-decreasing etc and a m. i. sequence is bounded below by its first term and a m. d. sequence is bounded above by its first term.

A sequence is a constant sequence iff it is m. i. as well as m. d.

A sequence which is m. i. or m. d. is called a monotone sequence.

1.4 Theorem. Let \( \{s_n\} \) be a m. i. sequence. If \( \{s_n\} \) is bounded then it converges to \( \sup_n s_n \), otherwise diverges to \(+\infty\).

Proof: Let \( \{s_n\} \) be a m. i. bounded sequence. Then the range \( \{s_1, s_2, s_3, \ldots\} \) of the sequence is bounded above and by lub axiom, its lub \( l \in \mathbb{R} \), i. e. \( \ell = \text{lub} \{s_1, s_2, s_3, \ldots\} = \sup_n s_n \).

For any \( \varepsilon > 0 \), \( l - \varepsilon \) is not an upper bound of \( \{s_1, s_2, \ldots\} \), so \( \exists n_0 \in \mathbb{N} \) such that \( l - \varepsilon < s_{n_0} \).

But \( s_{n_0} \leq s_{n_0 + 1} \leq s_{n_0 + 2} \leq \ldots \leq l + \varepsilon \). Hence \( \forall \varepsilon > 0 \), \( \exists n_0 \in \mathbb{N} \) such that \( n \geq n_0 \Rightarrow l - \varepsilon < s_n < l + \varepsilon \), i. e. \( |s_n - l| < \varepsilon \). This proves \( \lim_{n \to \infty} s_n = l = \sup s_n \).

If \( \{s_n\} \) is not bounded above, then for any \( M > 0 \), \( \exists n_0 \in \mathbb{N} \) such that \( s_{n_0} > M \).

Then \( \forall n \geq n_0 \Rightarrow s_n \geq s_{n_0} \Rightarrow s_n > M \), i. e. \( \lim_{n \to \infty} s_n = +\infty \). ■

Corollary 1. Let \( \{s_n\} \) a m. d. sequence. If \( \{s_n\} \) is bounded then it converges to \( \inf_n s_n \), otherwise diverges to \(-\infty\).

Subsequence: If \( \{k_n\}_{n=1}^{\infty} \) is any strictly increasing sequence of positive integers (i. e. \( k_n < k_{n+1} \) in \( \mathbb{N} \) and \( k_n \geq n \) \( \forall n \in \mathbb{N} \)) then \( \{s_{k_n}\}_{n=1}^{\infty} \) is a subsequence of \( \{s_n\}_{n=1}^{\infty} \).

1.5 Theorem. If \( \lim_{n \to \infty} s_n = l \) (or \( \pm \infty \)) then any subsequence of \( \{s_n\} \) converges to \( l \) (or \( \pm \infty \)).

Proof: (i) Let \( \lim_{n \to \infty} s_n = l \). Then \( \forall \varepsilon > 0 \), \( \exists n_0 \in \mathbb{N} \) such that \( n \geq n_0 \Rightarrow |s_n - l| < \varepsilon \).

Let \( \{s_{k_n}\}_{n=1}^{\infty} \) be any subsequence of \( \{s_n\} \). Then \( n \geq n_0 \Rightarrow k_n \geq n \geq n_0 \Rightarrow |s_{k_n} - l| < \varepsilon \).

Hence \( \lim_{n \to \infty} s_{k_n} = l \).

(ii) Let \( \lim_{n \to \infty} s_n = \infty \). Then \( \forall M > 0 \) (however large) \( \exists n_0 \in \mathbb{N} \) such that \( n \geq n_0 \Rightarrow s_n > M \Rightarrow s_{k_n} > M \) for any subsequence \( \{s_{k_n}\} \) of \( \{s_n\} \Rightarrow \lim_{n \to \infty} s_{k_n} = \infty \). ■

From above theorem (by contra positive): If a sequence has two subsequences with different limits then the sequence is not convergent.

1.6 Theorem. \( \lim_{n \to \infty} s_n = l \) (or \( \pm \infty \)) iff \( \lim_{n \to \infty} s_{2n} = \lim_{n \to \infty} s_{2n-1} = l \) (or \( \pm \infty \)).

Proof: Let \( \lim_{n \to \infty} s_n = l \), i. e. \( \{s_n\} \) is a convergent sequence and its limit is \( l \in \mathbb{R} \).

Then every subsequence of \( \{s_n\} \) also converges to \( l \). In particular \( \lim_{n \to \infty} s_{2n} = \lim_{n \to \infty} s_{2n-1} = l \).

Conversely let \( \lim_{n \to \infty} s_{2n} = \lim_{n \to \infty} s_{2n-1} = l \). Then \( \forall \varepsilon > 0 \), \( \exists n_0 \in \mathbb{N} \) such that \( n \geq n_0 \Rightarrow |s_{2n} - l| < \varepsilon \) and |s_{2n-1} - l| < \varepsilon. (*)

Consider any integer \( m \geq 2n \). If \( m \) is even, say \( m = 2n \), then \( 2n \geq 2n \Rightarrow n \geq n_0 \) and if \( m \) is odd, say \( m = 2n - 1 \), then \( 2n - 1 \geq 2n \Rightarrow n > n_0 \), so by (*) we have \( |s_m - l| < \varepsilon \) \( \forall m \geq 2n-1 \).

Hence \( \lim_{n \to \infty} s_{2n} = l \), i. e. \( \lim_{n \to \infty} s_n = l \). ■

Cauchy sequence. A sequence \( \{s_n\} \) is called a Cauchy sequence iff \( \forall \varepsilon > 0 \), \( \exists n_0 \in \mathbb{N} \) such that \( m, n \geq n_0 \Rightarrow |s_m - s_n| < \varepsilon \), i. e. \( n \geq n_0 \Rightarrow |s_{n+p} - s_n| < \varepsilon \) \( \forall p \in \mathbb{N} \).

Result: In \( \mathbb{R} \) (or \( \mathbb{C} \)), a sequence is Cauchy if it is convergent. Every convergent sequence is bounded but not conversely.

For example \( \{(−1)^{n−1}\}^{\infty}_{n=1} = (1, −1, 1, −1, 1, −1, \ldots) \) is bounded, has two subsequences \( (1, 1, 1, \ldots) \to 1, (1, −1, −1, \ldots) \to −1 \) with different limits and hence the sequence is not convergent.

Limit points: A number \( p \) is said to be a limit (cluster) point of a sequence \( \{s_n\} \) of every nbd of \( p \) contains infinitely many terms of the sequence.

In this case \( \forall \varepsilon > 0, s_n \in (p - \varepsilon, p + \varepsilon) \) for infinitely many values of \( n \). If \( c \) is the limit of a sequence \( \{s_n\} \) then \( c \) is the only limit point of \( \{s_n\} \).

For a bounded sequence of real numbers \( \lim_{n \to \infty} s_n \) and \( \lim_{n \to \infty} s_n \) are limit points, which are the least and greatest. The set of limit points of a bounded sequence is bounded and it has the greatest and least limit points. The set of limit points of a sequence is a closed set and hence the set of limit points of a bounded sequence is a compact set.

1.7 Theorem (Bolzano – Weierstrass Theorem): Every bounded sequence has a convergent subsequence.

B W Theorem also stated as: Every bounded sequence has at least one limit point.

1.8 Theorem. Every Cauchy sequence of real numbers is bounded.

Proof: Let \( \{s_n\} \) be a Cauchy sequence. Then for \( \varepsilon = 1, \exists k \in \mathbb{N} \) such that \( m, n \geq k \Rightarrow |s_m - s_n| = |s_m - s_k + s_k - s_n| \leq |s_m - s_k| + |s_k| < 1 + |s_k| \forall n \geq k \).

Let \( M = \max \{|s_1|, |s_2|, \ldots, |s_k|, 1 + |s_k|\} \). Then \( |s_n| \leq M \forall n \in \mathbb{N} \), i. e. \( \{s_n\} \) is a bounded sequence. ■

1.9 Theorem. Cauchy’s convergence criterion: A sequence \( \{s_n\} \) of real numbers converges iff \( \{s_n\} \) is a Cauchy sequence, i. e. \( \forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \) such that \( m, n \geq n_0 \Rightarrow |s_m - s_n| < \varepsilon \).
Above theorem shows that R is a complete metric space under the usual metric.

**Limit superior and limit inferior:**

Let \( \{s_n\} \) be a sequence of real numbers and for \( n \in \mathbb{N} \), let \( a_n = \text{glb} \{s_n, s_{n+1}, s_{n+2}, \ldots\} = \inf_{k \geq n} s_k \)
\( b_n = \text{lub} \{s_n, s_{n+1}, s_{n+2}, \ldots\} = \sup_{k \geq n} s_k \). Then \( a_n \leq s_n \leq b_n \) \( \forall n \in \mathbb{N} \).

\( \{a_n\} \) is m. i. and bounded above by \( b_1 \) (bounded below by \( a_1 \)), so \( \lim_{n \to \infty} a_n = \sup_{n \geq 2} a_n \in \mathbb{R} \) or \( \lim_{n \to \infty} a_n = +\infty \) (\( \lim_{n \to \infty} a_n = -\infty \) if \( a_n = -\infty \) \( \forall n \)).

\( \{b_n\} \) is m. d. and bounded above by \( a_1 \) (bounded above by \( b_1 \)), so \( \lim_{n \to \infty} b_n = \sup_{n \geq 2} b_n \in \mathbb{R} \) or \( \lim_{n \to \infty} b_n = +\infty \). Also note that \( \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n \).

\( \lim_{n \to \infty} a_n \) is called a lower limit or limit inferior of the sequence \( \{s_n\} \) and it is denoted by \( \lim inf s_n \) or \( \lim_{n \to \infty} s_n \).

\( \lim_{n \to \infty} b_n \) is called an upper limit of the sequence \( \{s_n\} \) and it is denoted by \( \lim sup s_n \) or \( \lim_{n \to \infty} s_n \).

Note that \( \lim_{n \to \infty} a_n = \ell \) \((\pm \infty)\) iff \( \lim sup s_n = \lim_{n \to \infty} s_n = \ell \) \((\pm \infty)\) iff \( \lim_{n \to \infty} s_{2n} = \lim_{n \to \infty} s_{2n+1} = \ell \) \((\pm \infty)\).

Limit superior, limit inferior of a sequence of real numbers always exist in the extended real number system.

Note that \( \lim_{n \to \infty} s_n = \lim_{n \to \infty} a_n = \sup_{n \geq 2} a_n \) and \( \lim_{n \to \infty} s_n = \inf_{n \geq 2} \sup_{k \geq n} s_k \).

For \( s_n \in \mathbb{R} \) \( \forall n \), let \( a = \lim_{n \to \infty} s_{2n}, b = \lim_{n \to \infty} s_{2n+1} \) \( \{s_n\} \) (if exist) then \( \lim s_n = \min \{a, b\}, \lim sup s_n = \max \{a, b\} \).

**Results:**

1. A real number \( u \) is the limit superior of a bounded sequence \( \{s_n\} \) iff (i) \( \forall \ v > 0, s_n > u - v \) \( \forall \text{ infinitely many values of } n \) and (ii) \( \forall \ v > 0, s_n < u + v \) \( \forall \text{ all except finitely many values of } n \).

2. A real number \( \ell \) is the limit inferior of a bounded sequence \( \{s_n\} \) iff (i) \( \forall \ v > 0, s_n < \ell + v \) \( \forall \text{ infinitely many values of } n \) and (ii) \( \forall \ v > 0, s_n > \ell - v \) \( \forall \text{ all except finitely many values of } n \).

3. A sequence \( \{s_n\} \) converges to \( \ell \) iff \( \lim sup s_n = \lim inf s_n = \ell \).

**1.10 Theorem:** Every sequence contains a monotonic subsequence.

**Proof:** If a sequence \( \{s_n\} \in \mathbb{N} \) has a term \( c \) repeated infinitely, then it has a constant subsequence \( \{c, c, c, \ldots\} \) of the sequence \( \{s_n\} \), which is monotone.

Next consider \( \{s_n\} \neq \infty \) as a sequence which has infinitely many distinct terms and no term is infinitely repeated. Let \( U = \lim sup s_n \) and \( L = \lim inf s_n \) be extended real numbers.

For \( U = \infty \), choose \( k_1 \in \mathbb{N} \) such that \( k_1 > 1 \). Choose least integer \( k_2 > k_1 \) such that \( s_{k_2} > \max \{2, s_{k_1}\} \) since \( s_n \in (\max \{2, s_{k_1}\}, \infty) \) for infinitely many values of \( n \), as \( \lim sup s_n = \infty \). Next choose integer \( k_3 > k_2 \) such that \( s_{k_3} > \max \{3, s_{k_2}\} \) and so on. In this way we obtain a strictly increasing subsequence \( \{s_{k_n}\} \) of \( \{s_n\} \) such that \( s_{k_n} > n \) \( \forall n \in \mathbb{N} \), showing \( \lim s_{k_n} = \infty \).

For \( L = -\infty \), we have \( \lim sup (-s_n) = -\lim inf s_n = -L = \infty \), so the sequence \( \{-s_n\} \) has strictly increasing subsequence \( \{-s_{k_n}\} \) diverging to \( +\infty \). \( \Rightarrow \{s_{k_n}\} \) has strictly decreasing subsequence \( \{s_{k_n}\} \) diverging to \( -\infty \). For \( U \in \mathbb{R}, s_n \in (U - \varepsilon, U) \) for infinitely many values of \( n \) or \( s_n \in (U, U + \varepsilon) \) for infinitely many values of \( n \). Consider \( s_n \in (U - \varepsilon, U) \) for infinitely many values of \( n \) and let \( A = \{s_n \mid s_n \in (U - \varepsilon, U)\} \). For any \( s_m \in A \), take \( s_m = s_{k_1} \), i.e. \( k_1 = m \). Choose least \( k_2 \in \mathbb{N}, k_2 > k_1 \) and \( s_{k_2} \in A \) with \( s_{k_2} > s_{k_1} \). Next choose least \( k_3 \in \mathbb{N}, k_3 > k_2 \) and \( s_{k_3} \in A \) with \( s_{k_3} > s_{k_2} \) and so on. In this way we obtain a monotonic decreasing subsequence \( \{s_{k_n}\} \) converging to \( U \). Similarly if \( s_n \in (U, U + \varepsilon) \) for infinitely many values of \( n \), we obtain a monotonic decreasing subsequence of \( \{s_{k_n}\} \) converging to \( U \). For \( L \in \mathbb{R}, s_n \in (L - \varepsilon, L) \) for infinitely many values of \( n \) or \( s_n \in (L, L + \varepsilon) \) for infinitely many values of \( n \). Consider \( s_n \in (L - \varepsilon, L) \) for infinitely many values of \( n \) and let \( B = \{s_n \mid s_n \in (L, L + \varepsilon)\} \). For any \( s_m \in B \), take \( s_m = s_{t_1} \), i.e. \( t_1 = m \). Choose least \( t_2 \in \mathbb{N}, t_2 > t_1 \) and \( s_{t_2} \in A \) with \( s_{t_2} > s_{t_1} \). Next choose least \( t_3 \in \mathbb{N}, t_3 > t_2 \) and \( s_{t_3} \in B \) with \( s_{t_3} > s_{t_2} \) and so on. In this way we obtain a monotonic increasing subsequence \( \{s_{k_n}\} \) converging to \( L \). Similarly if \( s_n \in (L - \varepsilon, L) \) for infinitely many values of \( n \), we obtain a monotonic increasing subsequence of \( \{s_{k_n}\} \) converging to \( L \).

Thus every sequence has a monotone subsequence.

**1.11 Theorem:** If a bounded sequence of real numbers has unequal \( \lim sup \) and \( \lim inf \) then it has at least two monotone (convergent) subsequences whose range sets are disjoint.

**Proof:** Let \( \{s_n\} \in \mathbb{N} \) be a bounded sequence of real numbers. Then \( \lim inf s_n = L, \lim sup s_n = U \) are real numbers and \( L \leq U \). Consider \( L \neq M, i.e. L < M \). Then \( \varepsilon = \frac{M - L}{3} > 0 \) is a real number.

(i) If \( L \) appears infinitely as terms of the sequence \( \{s_n\} \) then \( \{s_{k_n}\} = (L, L, L, \ldots) \) is a monotone subsequence (converging to \( L \)) of the sequence \( \{s_n\} \) and \( s_{k_n} \in (L - \varepsilon, L + \varepsilon) \) \( \forall n \in \mathbb{N} \).

Next consider that \( L \) appears as terms of the sequence \( \{s_n\} \) at most a finite number. Then \( s_n \in (L - \varepsilon, L) \) for infinitely many values of \( n \), in which case the sequence \( \{s_n\} \) has a monotone increasing subsequence.
\{s_n\}_{n=1}^{\infty} converging to L and \{s_k\}_{n=1}^{\infty} ∈ (L - \epsilon, L) ∀ n ∈ N; or \{s_n\} ∈ (L, L + \epsilon) for infinitely many values of n, in which case the sequence \{s_n\}_{n=1}^{\infty} has a monotone decreasing subsequence \{s_k\}_{n=1}^{\infty} converging to L and \{s_k\}_{n=1}^{\infty} ∈ (L, L + \epsilon) ∀ n ∈ N. Thus \{s_n\}_{n=1}^{\infty} has a monotone subsequence \{s_k\}_{n=1}^{\infty} converging to L and \{s_n\} ∈ (L - \epsilon, L + \epsilon) ∀ n ∈ N.

(ii) If U appears infinitely as terms of the sequence \{s_n\}_{n=1}^{\infty} then \{s_n\}_{n=1}^{\infty} = (U, U, \ldots) is a monotone subsequence (converging to U) of the sequence \{s_n\}_{n=1}^{\infty} and \{s_n\} ∈ (U - \epsilon, U + \epsilon) ∀ n ∈ N. Next consider that U appears as terms of the sequence \{s_n\}_{n=1}^{\infty} at most a finite number. Then \{s_n\} ∈ (U - \epsilon, U) for infinitely many values of n, in which case the sequence \{s_n\}_{n=1}^{\infty} has a monotone increasing subsequence \{s_{l+1}\}_{n=1}^{\infty} converging to U and \{s_n\} ∈ (U - \epsilon, U) ∀ n ∈ N; or \{s_n\} ∈ (U, U + \epsilon) for infinitely many values of n, in which case the sequence \{s_n\}_{n=1}^{\infty} has a monotone decreasing subsequence \{s_{l+1}\}_{n=1}^{\infty} converging to U and \{s_n\} ∈ (U, U + \epsilon) ∀ n ∈ N.

Thus \{s_n\}_{n=1}^{\infty} has a monotone subsequence \{s_{l+1}\}_{n=1}^{\infty} converging to U and \{s_n\} ∈ (U - \epsilon, U + \epsilon) ∀ n ∈ N. We have \{s_{l+1}\}_{n=1}^{\infty} is a monotone (convergent) subsequence of \{s_n\}_{n=1}^{\infty} in (L - \epsilon, L + \epsilon), \{s_n\}_{n=1}^{\infty} is a monotone (convergent) subsequence of \{s_n\}_{n=1}^{\infty} in (U - \epsilon, U + \epsilon) and (L - \epsilon, L + \epsilon)∩(U - \epsilon, U + \epsilon) = ∅. So we have at least two monotone subsequences \{s_{l+1}\}_{n=1}^{\infty} and \{s_{l+1}\}_{n=1}^{\infty} of the sequence \{s_n\}_{n=1}^{\infty} whose range sets are disjoint.

**Note.** From Theorem 1.9: (1) Every bounded sequence of real numbers has a monotone (and bounded) subsequence, which is convergent. Limit of this convergent subsequence, which is a real number, is a limit point of the sequence. From this B W theorem follows.

(2) Consider any Cauchy sequence \{s_n\}_{n=1}^{\infty} of real numbers. Then \{s_n\}_{n=1}^{\infty} is a bounded sequence and hence it has a bounded monotone subsequence \{s_{n_k}\}_{n=1}^{\infty}, which is convergent in \(R\). Then \(\exists \ l \in \ R \) with \(s_{n_k} \rightarrow l\) as \(n \rightarrow \infty\).

This implies \(s_n \rightarrow l\) as \(n \rightarrow \infty\), showing the Cauchy sequence \{s_{n_k}\}_{n=1}^{\infty} is convergent in \(R\) (with limit \(l \in \ R\) here). Thus every Cauchy sequence in \(R\) is convergent in \(R\).

Therefore \(R\) is a complete space.

**1.12 Theorem.** If \{s_{n_k}\}_{n=1}^{\infty} is a Cauchy sequence (or monotone) and it has a convergent subsequence \{s_{n_k}\}_{n=1}^{\infty} with \(s_{n_k} \rightarrow l\) as \(n \rightarrow \infty\), then \(s_n \rightarrow l\) as \(n \rightarrow \infty\).

**Proof:** Let \{s_{n_k}\}_{n=1}^{\infty} be a Cauchy sequence and it has a convergent subsequence \{s_{n_k}\}_{n=1}^{\infty} and \(\lim_{n \to \infty} s_{n_k} = l \in \ R\), for some \(l\).

As \(s_{n_k}\) is Cauchy, so \(\forall \ \epsilon > 0, \ \exists n_0 \in \ N\) such that \(m, n > n_0 \Rightarrow |s_m - s_n| < \epsilon/2\), and also \(\forall k_n > n_0\), \(|s_m - s_{n_k}| < \epsilon/2\). Taking \(n \to \infty\), \((k_n \geq n_0)\), and using \(s_{n_k} \to l\) as \(n \to \infty\), we get \(|s_m - l| < \epsilon\) \(\forall\) \(m \geq n_0\).

This proves \(\lim_{n \to \infty} s_n = l\). 

Proof of above theorems, results are available in [1] – [4] or in any standard books on Real Analysis and also further details on topics.

### 2. Euclidean Spaces

The concept of a metric space was originated in the Ph. D. thesis of Maurice Frechet presented to University of Paris in 1906. The definition of metric presently in use is given by German mathematician F. Hausdorff in 1914.

**Metric and Metric Space:** \(d\) is called a metric on a nonempty set \(X\) if \(d: X \times X \to \ R\) is a function such that

(i) \(d(x, y) \geq 0\) \(\forall x, y \in X\) and \(d(x, y) = 0\) \(\iff x = y\) \(\text{(Positivity)}\)

(ii) \(d(x, y) = d(y, x)\) \(\forall x, y \in X\) and \(\text{(Symmetry)}\)

(iii) \(d(x, y) \leq d(x, z) + d(z, y)\) \(\forall x, y, z \in X\). \(\text{(Triangle Inequality)}\)

A set \(X\) with a metric \(d\) on it, is called a metric space and it is denoted by \((X, d)\). Let \(\{s_n\}\) be a sequence in a metric space \((X, d)\).

(a) If \(l \in X\) is such that \(\forall \ \epsilon > 0, \ \exists n_0 \in \ N\) with \(n \geq n_0 \Rightarrow d(s_n, l) < \epsilon\) then we say that \(s_n\) is a convergent sequence and we can say that the sequence \(s_n\) converges to \(l\).

A sequence is said to be divergent if it is not convergent.

(b) Sequence \(\{s_{n_k}\}_{n=1}^{\infty}\) is said to be bounded if \(\exists \lim \ M > 0\) such that \(d(s_m, s_n) \leq M \ \forall \ m, n \in N\).

**Result 1.** If for any fixed \(c \in X\), \(\exists M > 0\) such that \(d(s_n, c) \leq M \ \forall n \in N\), then the sequence \(\{s_n\}_{n=1}^{\infty}\) in the metric space \((X, d)\) is bounded.

This follows from triangle inequality: \(d(s_m, s_n) \leq d(s_m, c) + d(s_n, c) \leq M + M = 2M \ \forall \ m, n \in N\).

(c) Sequence \(\{s_n\}\) is said to be a Cauchy sequence if \(\forall \ \epsilon > 0, \ \exists n_0 \in \ N\) such that \(m, n \geq n_0\) \(\Rightarrow d((x_m, x_n) < \epsilon);\) or equivalently \(n \geq n_0 \Rightarrow d((x_m, x_n) < \epsilon \ \forall \ p \in \ N\).

Note that every convergent sequence is Cauchy, but in general its converse is not true.
Result 2. Every Cauchy sequence in a metric space is bounded.

Proof: Let \((s_n)_{n=1}^{\infty}\) be a Cauchy sequence in a metric space \((X, d)\). Then for \(\varepsilon = 1, \exists p \in \mathbb{N}\) such that \(m, n \geq p \Rightarrow d(s_m, s_n) < 1\). Now \(L = \max \{d(s_i, s_j) : 1 \leq i, j < p\} \in \mathbb{R}\) and \(L \geq 0\). For any \(j < p\) and \(n \geq p\), we have \(d(s_j, s_p) \leq d(s_j, s_n) + d(s_p, s_n) < L + 1\). Then \(d(s_m, s_n) < L + 1\ \forall\ m, n \in \mathbb{N}\), showing that \((s_n)_{n=1}^{\infty}\) is a bounded sequence in \(X\).

(d) A metric space \(X\) is said to be a complete space if every Cauchy sequence in \(X\) is convergent in \(X\).

(e) \(d(x, y) = |x - y|\), for \(x, y \in \mathbb{R}\), is a metric, called the usual (Euclidean) metric on \(\mathbb{R}\).

(f) For any \(x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n\); (where \(m \in \mathbb{N}\) fixed), \(d(x, y) = \|x - y\| = (x_1^2 - y_1^2)^2 + (x_2^2 - y_2^2)^2 + \cdots + (x_n^2 - y_n^2)^2\) defines a metric on \(\mathbb{R}^n\), called the usual (or Euclidean) metric on \(\mathbb{R}^n\) and under this metric \(\mathbb{R}^n\) is called a Euclidean space.

Here \(\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\) is called norm of \(x\).

Note that \(|x| \leq \|x\| \leq \|y\| \forall x, y \in \mathbb{R}^n\).

A sequence \((S_n)\) in the Euclidean space \(\mathbb{R}^n\) is bounded if \(\exists a \in \mathbb{R}^n\) such that \(\|S_n\| \leq M, \forall n \in \mathbb{N}\). Here \(M = (0, 0, \ldots, 0) \in \mathbb{R}^n\).

2.1 Lemma: Let \(S_n = (a_n, b_n, \ldots, t_n) \in \mathbb{R}^m \forall n \in \mathbb{N}\), then \((S_n)\) is a bounded sequence in \(\mathbb{R}^m\) if all sequences \((a_n), (b_n), \ldots, (t_n)\) are bounded in \(\mathbb{R}\).

Proof: Let \((S_n)\) be a bounded sequence in \(\mathbb{R}^m\). Then \(\exists a \in \mathbb{R}^n\) such that \(\|S_n\| \leq M, \forall n \in \mathbb{N}\). All sequences \((a_n), (b_n), \ldots, (t_n)\) are bounded in \(\mathbb{R}\), conversely let all \(m\) sequences \((a_n), (b_n), \ldots, (t_n)\) in \(\mathbb{R}\) are bounded in \(\mathbb{R}\).

Then \(\exists K > 0, (\frac{K}{\sqrt{m}} > 0), \text{ such that } |a_n| \leq \frac{K}{\sqrt{m}}, |b_n| \leq \frac{K}{\sqrt{m}}, \ldots, |t_n| \leq \frac{K}{\sqrt{m}} \forall n \in \mathbb{N}\). Hence \((S_n)\) is a bounded sequence in \(\mathbb{R}^m\).

2.2 Theorem. Bolzano-Weierstrass Theorem: Every bounded sequence in a Euclidean space \(\mathbb{R}^m\) has a convergent subsequence. Proof: We prove the theorem using induction on \(m\).

We have: Every bounded sequence in \(\mathbb{R}\) has a convergent subsequence in \(\mathbb{R}\).

Theorem follows for the Euclidean space \(\mathbb{R}^m\) for \(m = 1\).

We prove the theorem for \(m = 2\): Let \((s_n)_{n=1}^{\infty}\) be any bounded sequence in \(\mathbb{R}^2\) and let \(s_n = (a_n, b_n) \forall n \in \mathbb{N}\), i.e. \(a_n, b_n \in \mathbb{R}\). Then \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are bounded sequences in \(\mathbb{R}\).

Hence both sequences have convergent subsequences in \(\mathbb{R}\). Let \((a_{kn})_{n=1}^{\infty}\) be a convergent subsequence of \((a_n)_{n=1}^{\infty}\) and \(l_1 \in \mathbb{R}\) be its limit; i.e. \(a_n \rightarrow l_1\) as \(n \rightarrow \infty\).

Then \((b_{kn})_{n=1}^{\infty}\) is a subsequence of the bounded sequence \((b_n)_{n=1}^{\infty}\) and hence bounded in \(\mathbb{R}\) and so it has a convergent subsequence in \(\mathbb{R}\), say \((b_{ktn})_{n=1}^{\infty}\).

So \(\exists l_2 \in \mathbb{R}\) such that \(b_{ktn} \rightarrow l_2\) as \(n \rightarrow \infty\). As any subsequence of a convergent sequence has the same limit that of the sequence, so the subsequence \((a_{ktn})_{n=1}^{\infty}\) of the convergent sequence \((a_{kn})_{n=1}^{\infty}\) has the limit \(l_1\).

Thus \(a_{ktn} \rightarrow l_1\) as \(n \rightarrow \infty\).

We have \((a_{ktn}, b_{ktn})_{n=1}^{\infty}\) as a subsequence of the sequence \((s_n)_{n=1}^{\infty}\) in \(\mathbb{R}^2\) and \((l_1, l_2) \in \mathbb{R}^2\), with \(\lim_{n \rightarrow \infty} a_{ktn} = l_1, \lim_{n \rightarrow \infty} b_{ktn} = l_2\). Then for any \(\varepsilon > 0, (\frac{\varepsilon}{\sqrt{2}} > 0), \exists n_0 \in \mathbb{N}\) such that

\[
n \geq n_0 \Rightarrow |a_{ktn} - l_1| < \frac{\varepsilon}{\sqrt{2}} \quad \text{and} \quad |b_{ktn} - l_2| < \frac{\varepsilon}{\sqrt{2}}
\]

\[
d((a_{ktn}, b_{ktn}), (l_1, l_2)) = \sqrt{(a_{ktn} - l_1)^2 + (b_{ktn} - l_2)^2} < \frac{\varepsilon^2}{2} = \varepsilon.
\]

This proves that any bounded sequence \((s_n)_{n=1}^{\infty}\) in \(\mathbb{R}^2\) has a convergent subsequence \((a_{ktn}, b_{ktn})_{n=1}^{\infty}\) in \(\mathbb{R}^2\) (with \(\lim_{n \rightarrow \infty} (a_{ktn}, b_{ktn}) = (l_1, l_2) \in \mathbb{R}^2\)).

Thus every bounded sequence in \(\mathbb{R}^2\) has a convergent subsequence in \(\mathbb{R}^2\), that is theorem is true for \(m = 2\). Let \(m > 2\) and assume that the theorem is true for \(m - 1\), i.e. every bounded sequence in \(\mathbb{R}^{m-1}\) has a convergent sequence in \(\mathbb{R}^{m-1}\). (Induction Hypothesis)
Let \( \{S_n\}_{n=1}^{\infty} \) be any bounded sequence in \( \mathbb{R}^2 \) and let \( S_n = (A_n, b_n) \forall n \in \mathbb{N} \), i.e. \( A_n \in \mathbb{R}^{m-1} \), \( b_n \in \mathbb{R} \). Then \( \{A_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) are bounded sequences in \( \mathbb{R}^m \), \( \mathbb{R} \) respectively. Hence both sequences have convergent subsequences in \( \mathbb{R}^m \) (by hypothesis), \( \mathbb{R} \) respectively. Let \( \{A_{kn}\}_{n=1}^{\infty} \) be a convergent subsequence of \( \{A_n\}_{n=1}^{\infty} \) and \( l \in \mathbb{R} \) be its limit; i.e. \( A_n \to l \) as \( n \to \infty \). Then \( \{b_{kn}\}_{n=1}^{\infty} \) is a subsequence of the bounded sequence \( \{b_n\}_{n=1}^{\infty} \) and hence bounded in \( \mathbb{R} \) and so it has a convergent subsequence in \( \mathbb{R} \), say \( \{b_{kn}\}_{n=1}^{\infty} \).

So if \( l \in \mathbb{R} \) such that \( b_{kn} \to l_2 \) as \( n \to \infty \). As any subsequence of a convergent sequence has the same limit that of the sequence, so the subsequence \( \{A_{kn}\}_{n=1}^{\infty} \) of the convergent sequence \( \{A_n\}_{n=1}^{\infty} \) has the limit \( l \). Thus \( A_{kn} \to l \) as \( n \to \infty \).

We have \( \{(A_{kn}, b_{kn})\}_{n=1}^{\infty} \) as a subsequence of the sequence \( \{S_n\}_{n=1}^{\infty} \) in \( \mathbb{R}^m \) and \((l, l_2) \in \mathbb{R}^m\), with \( \lim_{n \to \infty} A_{kn} = l \), \( \lim_{n \to \infty} b_{kn} = l_2 \). Then for any \( \varepsilon > 0 \), \( \exists n_0 \in \mathbb{N} \) such that

\[
\forall n \geq n_0 \implies \| A_{kn} - l \| < \frac{\varepsilon}{\sqrt{2}} \quad \text{and} \quad \| b_{kn} - l_2 \| < \frac{\varepsilon}{\sqrt{2}}.
\]

\[
\Rightarrow d((A_{kn}, b_{kn}), (l, l_2)) = \sqrt{(A_{kn} - l)^2 + (b_{kn} - l_2)^2} < \frac{\varepsilon^2 + \varepsilon^2}{2} = \varepsilon. \quad (E2)
\]

This proves that any bounded sequence \( \{S_n\}_{n=1}^{\infty} \) in \( \mathbb{R}^m \) has a convergent subsequence \( \{(A_{kn}, b_{kn})\}_{n=1}^{\infty} \) in \( \mathbb{R}^m \) (with \( (A_{kn}, b_{kn}) = (l, l_2) \in \mathbb{R}^m \)).

Hence the theorem follows by the first principle of mathematical induction. □

2.3 Lemma: Every Cauchy sequence in a Euclidean space is bounded.

**Proof:** Let \( \{s_n\}_{n=1}^{\infty} \) be any Cauchy sequence in a Euclidean space \( \mathbb{R}^m \).

Then for \( \varepsilon = 1, \exists p \in \mathbb{N} \) such that \( k, n \geq p \implies \|s_k - s_n\| < 1 \) \( \forall n \in \mathbb{N} \), i.e.

\[
\|s_n\| < \|s_p\| + 1 \quad \forall n \geq p.
\]

Let \( M = \text{max}\{\|s_1\|, \|s_2\|, \ldots, \|s_{p-1}\|, \|s_p\| + 1\} \).

Then \( M \in \mathbb{R}, M > 0 \) and \( \|s_n\| \leq M \forall n \in \mathbb{N} \). Hence the Cauchy sequence \( \{s_n\}_{n=1}^{\infty} \) is bounded.

2.4 Theorem: Every Euclidean space is complete, i.e. for each \( m \in \mathbb{N} \), \( \mathbb{R}^m \) is a complete metric space under the Euclidean metric.

**Proof:** Let \( \{s_n\}_{n=1}^{\infty} \) be any Cauchy sequence in \( \mathbb{R}^m \). Then \( \{s_n\}_{n=1}^{\infty} \) is a bounded sequence and hence it has a convergent subsequence, say \( \{s_{kn}\}_{n=1}^{\infty} \) and so \( s_{kn} \to l \) as \( n \to \infty \) for some \( l \in \mathbb{R}^m \). Now \( \{s_n\}_{n=1}^{\infty} \) is a Cauchy sequence, so \( \forall \varepsilon > 0 \), \( \exists n_0 \in \mathbb{N} \) such that

\[
t, n \geq n_0 \implies \|s_t - s_n\| < \varepsilon/2. \quad (E1)
\]

As \( k, n \geq n_0 \forall n \in \mathbb{N} \), since \( \{s_{kn}\}_{n=1}^{\infty} \) is a subsequence of the sequence \( \{s_n\}_{n=1}^{\infty} \), we have by (E1)

\[
n, t \geq n_0 \implies \|s_t - s_{kn}\| < \varepsilon/2. \quad (E2)
\]

Taking \( n \to \infty \) and using \( s_{kn} \to l \), we have by (E2),

\[
t, n \geq n_0 \implies \|s_t - l\| \leq \varepsilon/2.
\]

Thus \( \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \) such that \( n \geq n_0 \implies \|s_n - l\| < \varepsilon \), i.e. \( \lim_{n \to \infty} s_n = l \).

Hence \( \{s_n\}_{n=1}^{\infty} \) is a convergent sequence in \( \mathbb{R}^m \) with limit \( l \in \mathbb{R}^m \). □

**Conclusions**

The set of complex numbers \( \mathbb{C} \) is identified with \( \mathbb{R}^2 \) where \( z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C} \) are identified with \( z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in \mathbb{R}^2 \). Usual metric on \( \mathbb{C} \) is the Euclidean metric on \( \mathbb{R}^2 \), since \( |z_1 - z_2| = |(x_1 - x_2) + i(y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d((x_1, y_1), (x_2, y_2)) \).

Since \( \mathbb{R}^2 \) is complete, so \( \mathbb{C} \) is complete. As every bounded sequence in \( \mathbb{R}^2 \) has a convergent subsequence in \( \mathbb{R}^2 \), so every bounded sequence in \( \mathbb{C} \) has a convergent subsequence in \( \mathbb{C} \).

Also for any \( m \in \mathbb{N} \), \( \mathbb{R}^m \) is a complete space, so the unitary space \( \mathbb{C}^m \) (identified with \( \mathbb{R}^{2m} \)) is complete. Every bounded sequence in \( \mathbb{R}^m \) has a convergent subsequence, so every bounded sequence in \( \mathbb{C}^m \) has a convergent subsequence (B. W. Theorem).

**References**