A note on G-Complementary to Heron Mean

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ABSTRACT In this short note, the G-Complementary to Heron mean is introduced. A pair of double sequences in term of Heron means and G-Complementary to Heron mean are defined, their properties, monotonicity, Log-convexity and Log-concavity are discussed. Finally, as an illustration, it is justified that the new Gaussian compound mean $H_{2}(G)\otimes H_{c}$ converging faster than $H\otimes A$.

Keywords: Arithmetic mean, Geometric mean, Harmonic mean, Heron mean and G-Complementary Heron mean.

1. Introduction
For $p, q > 0$, some of the popular means well known in literature are Arithmetic mean $A(p, q) = \frac{p + q}{2}$, Geometric mean $G(p, q) = \sqrt{pq}$, Harmonic mean $H(p, q) = \frac{2pq}{p + q}$ and Heron mean $H_{e}(p, q) = \frac{p + \sqrt{pq} + q}{3}$, the various interesting results are found in [1, 5, 6-11, 15].

In [3, 12-14], researchers discussed about double sequences. As an illustration in [2], the popular iteration method called Herons’ iteration method is used to extract the square root of any positive number from Gaussian double sequences given by $p_{n+1} = H(p_{n}, q_{n})$ and $q_{n+1} = A(p_{n}, q_{n})$, also used the Archimedean double sequences $p_{n+1} = A(p_{n}, q_{n})$ and $q_{n+1} = G(p_{n+1}, q_{n})$ to get the approximate value of $\pi$. In [4], K. M. Nagaraja et al. discussed Logarithmic convexity and Logarithmic concavity of Archimedean and Gaussian double sequences. This work motivates to develop the paper.

2. Definitions and Results:
In this section, recall some definitions and lemmas which are essential for this paper.

Definition 2.1. [1, 15] A mean is put forth as a function $f = R_{+}^{2} \rightarrow R_{+}$ which has the property where $r \wedge s = \min(r, s)$ and $r \vee s = \max(r, s)$.

Definition 2.2. [15] A mean $N$ is called P-complementary to $M$ if it satisfies $P(M, N) = P$.

Suppose a given mean $M$ has an unique G-complementary mean $N$ is denoted by $N = M^{(G)} = \frac{G^{2}}{M}$

Then the G-complementary to Heron mean is defined as $H_{e}(G) = \frac{3rs}{r + \sqrt{rs} + s}$.

Definition 2.3. [3] The double sequences in terms of G-complementary to Heron mean and Heron mean are defined by

$r_{n+1} = H_{e}(G)(r_{n}, s_{n}) = \frac{3r_{n}s_{n}}{r_{n} + \sqrt{r_{n}s_{n}} + s_{n}}$ and $s_{n+1} = H_{e}(r_{n}, s_{n}) = \frac{r_{n} + \sqrt{r_{n}s_{n}} + s_{n}}{3}$. 

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Theorem 2.1. The Heron mean and G-complementary to Heron mean are related by the inequality

\[ p < H_e^{(G)}(p, q) < H_e(p, q) < q. \]

Proof: For \( p < q \), the Heron mean and G-complementary to Heron mean are given by

\[ H_e(p, q) = \frac{p + \sqrt{pq} + q}{3} \quad \text{and} \quad H_e^{(G)}(p, q) = \frac{3pq}{p + \sqrt{pq} + q}. \]

Consider

\[ H_e(p, q) - H_e^{(G)}(p, q) = \frac{(p + \sqrt{pq} + q)^2 - 9pq}{3(p + \sqrt{pq} + q)} = \frac{(p-q)^2 + 2\sqrt{pq}(\sqrt{p-q})^2}{3(p + \sqrt{pq} + q)} > 0. \]

Combining this inequality with lemma 2.1, gives \( p < H_e^{(G)}(p, q) < H_e(p, q) < q \).

Hence proof of Theorem 2.1 completes.

Theorem 2.2. For two distinct positive real values \( p_n < q_n \), the sequence \( p_{n+1} = H_e^{(G)}(p_n, q_n) \) is monotonically increasing and the sequence \( q_{n+1} = H_e(p_n, q_n) \) is monotonically decreasing. Also satisfy

\[ \text{Min}(p, q) = p = p_0 < p_1 < \ldots < p_n < p_{n+1} < q_{n+1} < q_n < \ldots < q_1 < q_0 = q = \text{Max}(p, q). \]

Proof: Let \( p_{n+1} = H_e^{(G)}(p_n, q_n) = \frac{3p_nq_n}{p_n + \sqrt{p_nq_n} + q_n} \) and \( q_{n+1} = H_e(p_n, q_n) = \frac{p_n + \sqrt{p_nq_n} + q_n}{3} \).

Consider

\[ \frac{p_{n+1}}{p_n} = \frac{3q_n}{\sqrt{p_n} + \sqrt{p_nq_n} + q_n} > \frac{3p_n}{p_n + \sqrt{p_nq_n} + p_n} = 1 \]

gives \( p_{n+1} > p_n \), which holds for all n. This proves that...
Similarly,
\[
\begin{align*}
q_{n+1} &= \sqrt[n]{q_n\left(\frac{p_n}{q_n} + \frac{q_n}{p_n}\right) + q_n} \\
&= \frac{q_n + \sqrt{p_n + q_n}}{\sqrt{3q_n}} < \frac{q_n + \sqrt{q_n + q_n}}{\sqrt{3q_n}} = 1
\end{align*}
\]

which holds for all \(n\). This proves that \(q_{n+1} < q_n\), which proves that \(q_{n+1} < q_n < q_{n-1} < \ldots < q_1 < q_0 = q = \text{Max}(p, q)\) \hspace{1cm} (2.2)

Eqs (2.1) and (2.2) lead to the proof of Theorem 2.2.

**Theorem 2.3.** For \(n \geq 0\), \(p_n < q_n\), the sequence \(p_{n+1} = H_e(G)(p_n, q_n)\) is Log-concave and the sequence \(q_{n+1} = H_e(p_n, q_n)\) is Log-convex.

Proof: If \(p_n < q_n\) the Heron mean G-complementary to Heron mean are given by
\[
q_{n+1} = H_e(p_n, q_n) = \frac{p_n + \sqrt{p_n q_n}}{3} \quad \text{and} \quad p_{n+1} = H_e(G)(p_n, q_n) = \frac{3p_n q_n}{p_n + \sqrt{p_n q_n} + q_n}
\]

Consider
\[
\frac{p_n}{p_{n+1}} - \frac{p_n - 1}{p_n} = \frac{\sqrt{p_n} - \sqrt{p_{n+1}}}{3q_n} > 0
\]

(\(\because \ p_n > p_{n-1} \quad \text{and} \quad -q_n > -q_{n-1}\))

So, \(p_n^2 > p_{n+1} p_{n-1}\) and hence \(p_{n+1} = H_e(G)(p_n, q_n)\) is log-concave.

Consider
\[
\frac{q_n}{q_{n+1}} - \frac{q_n - 1}{q_n} = \frac{3}{k}\left(q_n\left(\frac{p_n - 1}{q_n} + \frac{q_n - 1}{p_n}\right) - q_n\left(p_n + \sqrt{p_n q_n} + q_n\right)\right)
\]

Where \(k = \left(p_n + \sqrt{p_n q_n} + q_n\right)\left(p_n - 1\right) + \left(p_n q_n - q_n\right)\)

\[
< \frac{3}{k}\left(\sqrt{p_n - 1} - \sqrt{p_n}\right)\left(\sqrt{q_n - 1} + \sqrt{q_n}\right) \left(q_n + \sqrt{q_n q_n} - 1\right) < 0
\]

(\(\because \ p_n > p_{n-1} \quad \text{and} \quad q_n < q_{n-1}\))

So, \(q_n^2 < q_{n+1} q_{n-1}\) and hence \(q_{n+1} = H_e(p_n, q_n)\) is log-convex.

Hence the proof of Theorem 2.3 completes.

**Theorem 2.4.** The order \("(p_n)_{n \geq 0}\" \quad \text{and} \quad \"(q_n)_{n \geq 0}\"\) are defined in terms of G-complementary to Heron mean and Heron mean which are convergent to the common limit depicted as \(H_e(G) \otimes H_e(p, q) = G(p, q) = \sqrt{x}\).

Proof: We know that \(p_n < p_{n+1} < q_{n+1} < q_n\), \(n \geq 0\)

\[
q_{n+1} - p_{n+1} = \frac{\left(\sqrt{q_n - 1} - \sqrt{p_n}\right)^2}{3\left(p_n + \sqrt{p_n q_n} + q_n\right)} = \frac{\left(q_n - p_n\right)^2}{3}\left(p_n + \sqrt{p_n q_n} + q_n\right)
\]
Thus

\[ q_n - p_n < \frac{q - p}{3} \]

Which tends to 0 as \( n \to \infty \). Hence,

\[ \lim_{n \to \infty} p_n = \lim_{n \to \infty} q_n \]  

(2.4.1)

By Theorem 2.2, "\((p_n)_{n \geq 0}\) and \((q_n)_{n \geq 0}\)" are monotonically increasing and monotonically decreasing sequences respectively. Also,

\[ p_{n+1}q_{n+1} = H_e^{(g)}(p_n, q_n)H_e(p_n, q_n) = \left( \frac{3p_nq_n}{p_n + \sqrt{p_nq_n} + q_n} \right) \left( \frac{p_n + \sqrt{p_nq_n} + q_n}{3} \right) = p_nq_n \]

Therefore, \( p_{n+1}q_{n+1} = p_nq_n = p_{n-1}q_{n-1} = \ldots = p_0q_0 = x \)

Where 'x' is a multiple of two positive real numbers.

This implies that \( \lim_{n \to \infty} p_n = \lim_{n \to \infty} q_n = x \)

(2.4.2)

By equations (2.4.1) and (2.4.2), gives \( \lim_{n \to \infty} p_n = \lim_{n \to \infty} q_n = \sqrt{x} \)

Therefore the sequences "\((p_n)_{n \geq 0}\) and \((q_n)_{n \geq 0}\)" are convergent to a common limit \( \sqrt{x} \).

Hence the proof of Theorem 2.4 completes.

### 3. Application to Extracting Square Root

In (12), authors discussed Herons method of extracting square root using Gaussian compound mean \( H \otimes A \). In this section the convergence process of the new Gaussian compound mean mean \( H_e^{(g)} \otimes H_e \) converging faster than \( H \otimes A \) are discussed.

The following table 1 and figures (2.1) and (2.2) illustrate the approximate process of computing \( \sqrt{2} \). Also evident that \( H_e^{(g)} \otimes H_e \) is convergence to common limit faster than \( H \otimes A \).

<table>
<thead>
<tr>
<th>Gaussian Compound Mean</th>
<th>New Compound Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( H \otimes A )</td>
</tr>
<tr>
<td>( n )</td>
<td>( p_n )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
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<td>1.33333</td>
</tr>
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<td>1.414211</td>
</tr>
<tr>
<td>4</td>
<td>1.414214</td>
</tr>
<tr>
<td>5</td>
<td>...</td>
</tr>
</tbody>
</table>

Table-1: The values of Gaussian compound mean and New compound mean

![Figure 2.1: Comparison of Gaussian compound mean and New compound mean](image-url)
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References
15. G. Toader and S. Toader, Greek means and Arithmetic mean and Geometric mean, RGMIA Monograph,(2005), Australia.