THE EDGE-TO-EDGE MONOPHONIC NUMBER OF A GRAPH

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ABSTRACT In this paper we introduce the edge-to-edge monophonic number, m_{ee}(G) of a connected graph with at least 3 vertices and study some of its general properties. We also determine the edge-to-edge monophonic number of certain classes of graphs. For each pair of integers k and q with 2 ≤ k ≤ q, there exists a connected graph G of order q + 1 and size q with $m_{ee}(G) = k$. Connected graphs of size q ≥ 4 with edge-to-edge monophonic number q is characterized.

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1. INTRODUCTION

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. We consider connected graphs with at least three vertices. For basic graph theory terminology we refer to Harary [1]. For two vertices $u$ and $v$ in a connected graph $G$, the distance $d(u, v)$ is the length of a shortest $u – v$ path in $G$. An $u – v$ path of length $d(u, v)$ is called an $u – v$ geodesic. For a vertex $v$ of $G$, the eccentricity $e(v)$ is the distance between $v$ and a vertex farthest from $v$. The minimum eccentricity among the vertices is the radius, $rad G$ and the maximum eccentricity is the diameter, $diam G$ of $G$. For subsets $A$ and $B$ of $V(G)$, the distance $d(A, B)$ is defined as $d(A, B) = \min\{d(x, y) : x \in A, y \in B\}$. An $u – v$ path of length $d(A, B)$ is called an $A – B$ geodesic joining the sets $A, B$, where $u \in A$ and $v \in B$. A vertex $x$ is said to lie on an $A – B$ geodesic if $x$ is a vertex of an $A – B$ geodesic. For $A = (u, v)$ and $B = (z, w)$ with $uw$ and $zw$ edges, we write an $A – B$ geodesic as $uw – zw$ geodesic and $d(A, B)$ as $d(uw, zw)$. The maximum degree of $G$, denoted by $\Delta(G)$, is given by $\Delta(G) = \max\{\deg G(v) : v \in V(G)\}$, $N(v) = \{u \in V(G) : uv \in E(G)\}$ is called the neighborhood of the vertex $v$ in $G$. A vertex $v$ is an extreme vertex of a graph $G$ if the subgraph induced by its neighbors is complete. An edge $e$ of a graph $G$ is called an extreme edge of $G$, if one of its ends is an extreme vertex of $G$. A chord of a path $p_u, u_1, u_2, \ldots, u_k$ is an edge $u_i u_j$, with $j \geq i + 2$. An $u – v$ path is called a monophonic path if it is a chordless path. A monophonic set of $G$ is a set $M \subseteq V$ such that every vertex of $G$ lies on a monophonic path joining some pair of vertices in $M$. The monophonic number $m(G)$ of $G$ is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a minimum monophonic set or simply a $m$-set of $G$. The monophonic number of a graph is studied in [4, 6]. For a cut-vertex $u$ in a connected graph $G$ and a component $H$ of $G – u$, the sub graph $H$ and the vertex $u$ together with all edges joining $u$ and $V(H)$ is called a branch of $G$ at $u$.

The following theorems are used in sequel.

Theorem: 1.1.[6] Every end-edge of a connected graph $G$ belongs to every edge-to-vertex monophonic set of $G$.

Theorem: 1.2.[6] For a connected graph $G$ with $q \geq 4m_{ee}(G) = q$ if and only if $G = K_{1, q}$.

2. THE EDGE-TO-EDGE MONOPHONIC NUMBER OF A GRAPH

Definition 2.1. Let $G = (V, E)$ be a connected graph with at least 3 vertices. A set $M \subseteq E$ is called an edge-to-edge monophonic set of $G$ if every edge of $G$ lies on a monophonic path joining a pair of edges of $M$. The edge-to-edge monophonic number $m_{ee}(G)$ of $G$ is the minimum cardinality of its edge-to-edge monophonic sets and any edge-to-edge monophonic set of cardinality $m_{ee}(G)$ is said to be an $m_{ee}$-set of $G$. 

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Example 2.2. For the graph $G$ given in Figure 2.1, $M = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a $m_{ee}$-set of $G$ so that $m_{ee}(G)=3$.

Remark 2.3. There can be more than one $m_{ee}$-set of $G$ for the graph $G$ given in Figure 2.1, $M_1 = \{v_1, v_2, v_3, v_6, v_4, v_5\}$.

In the following we determine the edge-to-edge monophonic number of some standard graphs.

Theorem 2.4. For a connected graph $G$ of size $q \geq 2$, $2 \leq m_{ee}(G) \leq q$.

Proof. A $m_{ee}$-set wants at least two edgeto form monophonic path and therefore $m_{ee}(G) \geq 2$. Also, the set of all edges of $G$ is an edge-to-edge monophonic set of $G$ so that $m_{ee}(G) \leq q$. Thus $2 \leq m_{ee}(G) \leq q$.

Theorem 2.5. Let $v$ be an extreme vertex of a connected graph $G$, then every edge-to-edge monophonic set contains at least one extreme edge that is incident with $v$.

Proof. Let $v$ be an extreme vertex of a connected graph $G$. Let $e_1, e_2, ..., e_k$ be the edges incident with $v$. Let $M$ be any edge-to-edge monophonic set of $G$. We assume that $e_i \in M$ for all $i (1 \leq i \leq k)$. Suppose, $e_i \notin M$ for some $i (1 \leq i \leq k)$. Since $M$ is an edge-to-edge monophonic path and $e = u_hv_{h+2}$ lies on a monophonic path joining two elements say $x, y \in M$. Let $x = v_1v_2$ and $y = v_kv_m$. Then $v \neq v_1, v_2, v_kv_m$ and $d_m(x, y) \geq 1$. Assume that without loss of generality let $P : u_0 = v_1, u_1, u_2, ..., u_h, u_{h+1}, u_{h+2}, ..., u_{k-1}, u_k = vy$ be a $x - y$ monophonic path, where $u_1 \neq v_2$ and $u_{k-1} \neq v_m$. Since $v$ is an extreme vertex, $u_h$ and $u_{h+2}$ are adjacent and so $Q : u_0 = v_1, u_1, u_2, ..., u_h, u_{h+2}, u_{h+3}, ..., u_{k-1}, u_k = vh$ has a chord in $P$, which is a contradiction. Hence $e \in M$ for any $i (1 \leq i \leq k)$.

Corollary 2.6. Every edge of a connected graph $G$ belongs to every edge-to-edge monophonic set of $G$.

Proof. This follows from Theorem 2.5. ■

Theorem 2.7. Let $G$ be a connected graph with cut vertices and $M$ is an edge-to-edge monophonic set of $G$. Then every branch of $G$ contains an element of $M$.

Proof. Let us assume that there is a branch $B$ of $G$ at a cut-vertex $v$ such that $B$ contains no element of $M$. Then by Corollary 2.6, $B$ does not contain any end-vertex of $G$. Hence it follows that no vertex of $B$ is an end vertex of $G$. Let $u = xv$ be any edge of $B$ such that $z \neq x$ and $z \neq y$ (such a vertex exists since $|V(B)| \geq 2$). Then $u$ is not an edge of $M$ and so $u$ lies on a $e-f$ monophonic path $P : x_1, x_2, ..., x_j, y, ..., x_0$, where $x_1$ is an end of $e$ and $x_0$ is an end of $f$ with $e, f \in M$. Since $v$ is a cut-vertex of $G$, the $x_1-x$ and $x-x_0$ subpaths of $P$ both contain $v$ and so $P$ is not a path, which is a contradiction. Hence every branch of $G$ contains an element of $M$.

Corollary 2.8. Let $G$ be a connected graph with cut-edges and $M$ an edge-to-edge monophonic set of $G$. Then for any non-pendant cut-edge of $G$, each of the two components of $G - e$ contains an element of $M$.

Proof. Let $e = uv$. Let $G_1$ and $G_2$ be the two components of $G - e$ such that $u \in V(G_1)$ and $v \in V(G_2)$. Since $u$ and $v$ are cut-vertices of $G$, it follows that $G_1$ contains at least one branch at $u$ and $G_2$ contains at least one branch at $v$. Hence it follows from Theorem 2.7 that each of $G_1$ and $G_2$ contains an element of $M$.

Theorem 2.9. Let $G$ be a connected graph and $M$ be a $m_{ee}$-set of $G$. Then no non-pendant cut-edge of $G$ belongs to $M$. ■

Figure 2.1
Proof. Let $M$ be a $m_{ee}$-set of $G$. Suppose that $e = xy$ be a non-pendant cut-edge of $G$ such that $e 
otin M$. Let $G_1$ and $G_2$ be the two components of $G - e$. We claim that $M'$ is an edge-to-edge monophonic set of $G$. By Corollary 2B, $G_1$ contains an edge $uv$ and $G_2$ contains an edge $u'v'$, where $uv, u'v' \in M$. Let be any edge of $G$. Assume without loss of generality that $f$ belongs to $G_1$. Since $xy$ is a cut-edge of $G$, every monophonic path joining an edge of $G_1$ with an edge of $G_2$ contains the edge $xy$. Suppose that $w$ is incident with $xy$ or the edge $uv$ of $M$ or that lies on a monophonic path joining $uv$ and $xy$. If $f$ is adjacent to $xy$, then let $P = v_1, v_2, ..., v'$ be a $uf-xy$ monophonic path. Thus lies on the $uv-u'v'$ monophonic path $If f = uv$, then there is nothing to prove. If $f$ lies on an $xymonophonic path say $v_1, v_2, ..., v'$, then let $yv_1, yv_2, ..., yv'$ be a $uv-u'v'$ monophonic path. Then clearly $v_1, v_2, ..., yv_1, yv_2, ..., yv'$ is a $uv-u'v'$ monophonic path. Thus lies on a monophonic path joining a pair of edges of $M'$. Thus we have proved that any edge of $G$ is either lies on $M$ on that lies in a monophonic path joining $xy$ and $uv$ of $M$ and also lies on $M'$ or lies on a monophonic path joining a pair of edges of $M'$. Hence it follows that $M$ is an edge-to-edge monophonic path such that $|M'| = |M| - 1$, which is a contradiction to $M$ a $m_{ee}$-set of $G$. Hence the theorem follows.

**Theorem 2.10.** For any non-trivial tree $T$ with $k$ end-edge, $m_{ee}(T) = k$ the set of all end-edges of $T$ is the unique minimum edge-to-edge monophonic set of $T$.

**Proof.** This follows from Corollary 2.6 and Theorem 2.9.

**Theorem 2.11.** For the cycle $G = C_p(p \geq 4)$, $m_{ee}(G) = 2$.

**Proof.** Let $M = \{e, f\}$ be the set of two adjacent edges in $G$. Then $M$ is an edge-to-edge monophonic set of $G$ and so that $m_{ee}(G) = 2$.

**Theorem 2.12.** For the complete graph $G = K_p(p \geq 4)$, $m_{ee}(G) = \begin{cases} \frac{p}{2} & \text{if } p \text{ is even} \\ \frac{p+1}{2} & \text{if } p \text{ is odd} \end{cases}$

**Proof.** Let $p$ be even. Let $M$ be the set of $p/2$ independent edges of $G$. Then $M$ is an edge-to-edge monophonic set of $G$ and $som_{ee}(G) \leq p/2$. We prove that $m_{ee}(G) = p/2$. If not let $m_{ee}(G) < p/2$. Then there exists an edge-to-edge monophonic set $M'$ of $K_p$ such that $|M'| < p/2$. Then there exists at least one edge $e$ of $M$ such that $e \notin M'$. If $M'$ is the set of independent edges of $G$, then $e$ does not lie on a monophonic joining a pair of edges of $M'$. If $M'$ is not independent, then also $e$ does not lies on a monophonic joining a pair of edges of $M'$. Hence $M'$ is not an edge-to-edge monophonic set of $G$. which is a contradiction. Thus $M$ is a minimum edge-to-edge monophonic set. Therefore $m_{ee}(G) = p/2$.

Let $p$ be odd. Let $M$ be the set of $(p-1)/2$ independent edges and one adjacent edges of $G$. Then $M$ is an edge-to-edge monophonic set of $G$ and $som_{ee}(G) \leq (p-1)/2 + 1 = \frac{p+1}{2}$. We show that $m_{ee}(G) = \frac{p+1}{2}$. If not, let $m_{ee}(G) < \frac{p+1}{2}$. Then there exists at least one $e \in M$ such that $e \notin M'$. Hence $M'$ is not an edge-to-edge monophonic set of $G$ which is a contradiction. Thus $M$ is a minimum edge-to-edge monophonic set. Therefore $m_{ee}(G) = \frac{p+1}{2}$.

**Theorem 2.13.** For the complete bipartite graph $G = K_{m,n}(2 \leq m < n)$, $m_{ee}(G) = 2$.

**Proof.** Let $U = \{u_1, u_2, ..., u_m\}$ and $V = \{v_1, v_2, ..., v_n\}$ be a partition of $G$. Let $M = \{u_1, v_1, u_m, v_n\}$. Then every edge of $G$ lies on $u_1v_1 - u_mv_n$ monophonic path and so $M$ is a monophonic set of $G$. Therefore $m_{ee}(G) = 2$.

**Theorem 2.14.** Let $G$ be a connected graph which is not $C_3$ or a star. Then $m_{ee}(G) \leq q - 1$.

**Proof.** Let $e$ be an edge of $G$ which is not an end edge of $G$. Let $M = E(G) - \{e\}$. Since $e$ is not an end edge of $G$, $G$ is not a star. Since $G \neq C_3$, $M$ is an edge-to-edge monophonic set of $G$ so that $m_{ee}(G) \leq |M| = q - 1$.

**Theorem 2.15.** For each pair of integers $k$ and $q$ with $2 \leq k \leq q$, there exists a connected graph $G$ of order $q + 1$ and size $q$ with $m_{ee}(G) = k$.

**Proof.** For $2 \leq k \leq q$, let $P$ be a path of order $q - k + 3$. Let $G$ be the graph obtained from $P$ by adding $k - 2$ new vertices to $P$ and joining them to any cut-vertex of $P$. Clearly, $G$ is a tree of order $q + 1$ and size $q$ with $k$ end-edges and so by Theorem 2.10, $m_{ee}(G) = k$.

**REFERENCES**

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