THE EDGE-TO-EDGE MONOPHONIC NUMBER OF A GRAPH

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Received: July 02, 2018 Accepted: August 13, 2018

ABSTRACT *In this paper we introduce the edge-to-edge monophonic number* $m_{ee}(G)$ *of a connected graph with at least 3 vertices and study some of its general properties. We also determine the edge-to-edge monophonic number of certain classes of graphs. For each pair of integers k and q with* $2 \leq k \leq q$, *there exists a connected graph G of order* $q+1$ *and size q with* $m_{ee}(G)$= *k. Connected graphs of size* $q \geq 4$ *with edge-to-edge monophonic number q is characterized.*

Keywords*: monophonic path, monophonic number, edge monophonic number, edge-to-edge monophonic number.*
AMS Subject Classification: 05C12

1. INTRODUCTION

By a *graph* $G = (V, E)$we mean a finite undirected connected graph without loops or multiple edges.The order and size of G are denoted by$p$and$q$ respectively. We consider connected graph with at least three vertices. For basic graph theoretic terminology we refer to Harary [1]. For two vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in G. An$u - v$ path of length $d\ (u, v)$is called an $u - v$ *geodesic*. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices is the *radius, rad* G and the maximum eccentricity is the *diameter, diam* G of G. For subsets A and B of $V\ (G)$, the *distance* $d(A, B)$ is defined as $d(A, B) = min\{d(x, y) : x \in A, y \in B\}$. An$u - v$ path of length $d\ (A, B)$ is called an $A - B$ *geodesic* joining the sets A, B, where $u \in A$ and $v \in B$. A vertex x is said to lie on an $A - B$ *geodesic* if x is a vertex of an $A - B$ *geodesic*. For $A = (u, v)$ and $B = (z, w)$ with uv and zw edges, we write an $A - B$geodesic as $uv - zw$ *geodesic* and $d(A, B)$ as $d(uv, zw)$. The *maximum degree* of G, denoted by*Δ(G),* is given by $\Delta(G) = max\{degG(v): v \in V(G)\}, N(v) = \{u \in V(G): uv \in E(G)\}$ is called the *neighborhood* of the vertex v in G. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete. An edge e of a graph G is called an *extreme edge* of G, if one of its ends is an extreme vertex of G. A *chord* of a path$u_0, u_1, u_2, \ldots u_h$ is, an edge $u_i u_j$, with $j \geq i + 2$. An$u - v$ path is called a *monophonic path* if it is a chordless path. A *monophonic set* of G is a set $M \subseteq V$ such that every vertex of G lies on a monophonic path joining some pair of vertices in M. The monophonic number $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a minimum monophonic set or simply a *m-set* of G. The monophonic number of a graph is studied in [4, 6]. For a *cut – vertex v* in a connected graph G and a component $H$of $G - v$, the sub graph H and the vertex v together with all edges joining $v$and $V\ (H)$is called a *branch* of G at v.
The following theorems are used in sequel.

Theorem: 1.1.[6] Every end-edge of a connected graph G belongs to every edge -to-vertex monophonic set of G.

Theorem: 1.2.[6] For a connected graph G with $q \geq 4$,$m_{ev}(G) = q$ if and only if $G = K_{1,q}$.

2. THE EDGE-TO-EDGE MONOPHONIC NUMBER OF A GRAPH

Definition 2.1. Let $G = (V, E)$ be a connected graph with at least 3 vertices. A set $M \subseteq E$ is called an *edge-to-edge monophonicset*of G if every edge of G lies on a monophonic path joining a pair of edges of M. The *edge-to-edge monophonic number*$m_{ee}(G)$of G is the minimum cardinality of its edge-to- edge monophonic sets and any edge-to- edge monophonic set of cardinality $m_{ee}(G)$is said to be a m_{ee}-set of G.

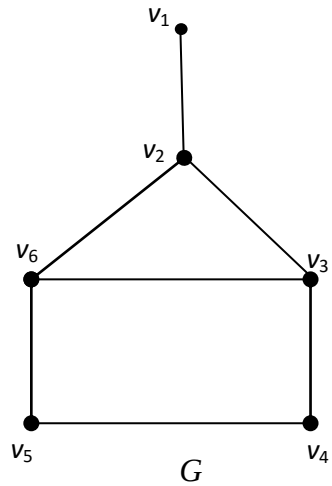

Figure 2.1

Example 2.2. For the graph G given in Figure 2.1, $M = \{v_1v_2, v_3v_4, v_5v_6\}$is a m_{ee}-set of G so that$m_{ee}(G)$=3.
Remark 2.3. There can be more than one m_{ee}-set of G for the graph G given in Figure 2.1,
M_1= $\{v_1v_2, v_3v_6, v_4v_5\}$
In the following we determining the edge-to-edge monophonic number of some standard graphs.
Theorem: 2.4. For a connected graph G of size $q \geq 2$, $2 \leq m_{ee}(G) \leq q$.
Proof.A m_{ee}-set wants at least two edgesto form monophonic path and therefore $m_{ee}(G) \geq 2$. Also, the set of all edges of G is an edge-to-edge monophonic set of G so that $m_{ee}(G) \leq q$. Thus $2 \leq m_{ee}(G) \leq q$. ■
Theorem: 2.5. Let$v$ be an extreme vertex of a connected graph G, then every edge-to-edge monophonic set contains at least one extreme edge that is incident with v.
Proof. Let v be an extreme vertex of a connected graph G. Let $e_1, e_2, ..., e_k$ be the edges incident with v. Let M be any edge-to-edge monophonic set of G. We assume that $e_i \in M$ for all $i(1 \leq i \leq k)$. Suppose, $e_i \notin M$for some$i$ $(1 \leq i \leq k)$. Since M is an edge-to-edge monophonic path and $e_i = u_{h+1}u_{h+2}$ lies on a monophonic path joining two elements say $x, y \in M$. Let $x = v_1v_2$ and $y = v_lv_m$. Then $v \neq v_1, v_2, v_l, v_m$ and $d_m(x, y) \geq 1$. Assume that without loss of generality let$P$: $u_0 = v_1, u_1, u_2, ..., u_h, u_{h+1}, u_{h+2}, ..., u_{t-1}, u_t = v_l$be a x –y monophonic path, where $u_1 \neq v_2$ and $u_{t-1} \neq v_m$.Since v is an extreme vertex, u_h and u_{h+2} are adjacent and so Q :$u_0 = v_1, u_1, u_2, ..., u_h, u_{h+2}, u_{h+3}, ..., u_{t-1}, u_t = v_l$has a chord in P, which is a contradiction. Hence $e_i \in M$ for any$i$ $(1 \leq i \leq k)$. ■
Corollary:2.6. Every end edge of a connected graph G belongs to every edge-to-edge monophonic set of G.
Proof. This follows from Theorem: 2.5.■
Theorem:2.7. Let G be a connected graph with cut vertices and M is an edge-to-edge monophonic set of G. Then every branch of G contains an element of M.
Proof. Let us assume that there is a branch B of G at a cut-vertex v such that B contains no element of M. Then by Corollary 2.6, B does not contain any end-edge of G. Hence it follows that no vertex of B is an end vertex of G. Let $u = xy$be any edge of B such that $z \neq x$and $z \neq y$(such a vertex exists since $|V(B)| \geq 2$). Then u is not an edge of M and so u lies on a e –$f$monophonic path$P$: $x_1, x_2, ..., x, y, ..., x_t$, where x_1 is an end of e and x_t is an end of f with $e, f \in M$. Since v is a cut-vertex of G, the x_1–x and$x$–$x_t$subpaths of P both contain v and so P is not a path, which is a contradiction. Hence every branch of G contains an element of M.■
Corollary:2.8. Let G be a connected graph with cut-edges and M an edge-to- edge monophonic set of G. Then for any non-pendant cut-edge e of G, each of the two components of $G - e$ contains an element of M.
Proof. Let $e = uv$. Let G_1 and G_2 be the two components of $G - e$ such that $u \in V(G_1)$ and $v \in V(G_2)$. Since u and v are cut-vertices of G, it follows that G_1 contains at least one branch at u and G_2 contains at least one branch at v. Hence it follows from Theorem 2.7 that each of $G_1$and G_2 contains an element of M. ■
Theorem: 2.9. Let G be a connected graph and M be a m_{ee}-set of G. Then no non-pendant cut-edge of G belongs to M.

Proof. Let M be a m_{ee} set of G. Suppose that $e = xy$ be a non-pendant cut-edge of G such that $e \in M$. Let G_1 and G_2 be the two components of $G - e$. Let $M' = M - \{xy\}$. We claim that M' is an edge-to-edge monophonic set of G. By Corollary 2.8, G_1 contains an edge uv and G_2 contains an edge $u'v'$, where $uv, u'v' \in M$. Let f be any edge of G. Assume without loss of generality that f belongs to G_1. Since xy is a cut edge of G, every monophonic path joining an edge of G_1 with an edge of G_2 contains the edge xy. Suppose that w is incident with xy or the edge uv of M or that lies on a monophonic path joining uv and xy. If f is adjacent to xy, then let $P: v, v_1, v_2, \ldots, f = x$ be a uv–f monophonic path. Let $Q: y, y_1, y_2, \ldots, v'$ be a y–$u'v'$ monophonic path. Then, it is clear that $P \cup \{xy\} \cup Q$ is a uv–$u'v'$ monophonic path. Thus f lies on the uv–$u'v'$ monophonic path. If $f = uv$, then there is nothing to prove. If f lies on a uv–xy monophonic path say $v, v_1, v_2, \ldots, f, \ldots, x$, then let $y, y_1, y_2, \ldots, v'$ be xy–$u'v'$ monophonic path. Then clearly $v, v_1, v_2, \ldots, f, \ldots, x, y, y_1, y_2, \ldots, v'$ is a uv–$u'v'$ monophonic path. Thus f lies on a monophonic path joining a pair of edges of M'. Thus we have proved that any edge of G is either lies on M on that lies in a monophonic path joining xy and uv of M and also lies on M' or lies on a monophonic path joining a pair of edges of M'. Hence it follows that M' is an edge-to-edge monophonic path such that $|M'| = |M| - 1$, which is a contradiction to M a m_{ee}-set of G. Hence the theorem follows. ■

Theorem: 2.10. For any non-trivial tree T with k end-edge, $m_{ee}(T) = k$ the set of all end-edges of T is the unique minimum edge-to- edge monophonic set of T.

Proof. This follows from Corollary: 2.6 and Theorem: 2.9. ■

Theorem: 2.11. For the cycle $G = C_p (p \geq 4)$, $m_{ee}(G) = 2$.

Proof. Let M={e,f} be the set of two adjacent edges in G. Then M is an edge-to-edge monophonic set of G and so that $m_{ee}(G) = 2$. ■

Theorem:2.12. For the complete graph $G = K_p (p \geq 4)$, $m_{ee}(G) = \begin{cases} \frac{p}{2} & if\ p\ is\ even \\ \frac{p+1}{2} & if\ p\ is\ odd \end{cases}$

Proof. Let p be even. Let M be the set of $p/2$ independent edges of G. Then M is an edge- to- edge monophonic set of G and so $m_{ee}(G) \leq p/2$. We prove that $m_{ee}(G) = p/2$. If not let $m_{ee}(G) < p/2$. Then there exists an edge-to-edge monophonic set M' of K_p such that $|M'| < p/2$. Then there exists at least one edge e of M such that $e \notin M'$. If M' is the set of independent edges of G, then e does not lie on a monophonic joining a pair of edges of M'. If M' is not independent , then also e does not lies on a monophonic joining a pair of edges of M'. Hence M' is not an edge- to-edge monophonic set of G. which is a contradiction. Thus M is a minimum edge- to-edge monophonic set. Therefore $m_{ee}(G) = p/2$.

Let p be odd. Let M be the set of $(p-1)/2$ independent edges and one adjacent edges of G. Then M is an edge- to- edge monophonic set of G and so $m_{ee}(G) \leq \frac{p-1}{2} + 1 = \frac{p+1}{2}$. We show that $m_{ee}(G) = \frac{p+1}{2}$. If not , let $m_{ee}(G) < \frac{p+1}{2}$. Then there exists at least one e∈M such that $e \notin M'$. Hence M' is not an edge- to-edge monophonic set of G.which is a contradiction. Thus M is a minimum edge-to-edge monophonic set. Therefore $m_{ee}(G) = \frac{p+1}{2}$.■

Theorem : 2.13. For the complete bipartite graph $G = K_{m,n} (2 \leq m < n)$, $m_{ee}(G) = 2$.

Proof: Let $U = \{u_1, u_2, \ldots u_m\}$ and $V = \{v_1, v_2, \ldots v_n\}$ be a bipartition of G. Let $M = \{u_1 v_1, u_m v_n\}$. Then every edge of G lies on $u_1 v_1 - u_m v_n$ monophonic path and so M is a monophonic set of G. Therefore $m_{ee}(G) = 2$.■

Theorem: 2.13. Let G be a connected graph which is not C_3 or a star. Then $m_{ee}(G) \leq q - 1$

Proof: Let e be an edge of G which is not an end edge of G. Let $M = E(G) - \{e\}$. Since e is not an end edge of G, G is not a star. Since $G \neq C_3$, M is a edge-to-edge monophonic set of G so that $m_{ee}(G) \leq |M| = q - 1$■

Theorem : 2.14. For the connected graph G, $m_{ee}(G) = q$ if and only if $G = K_{1,q}$ or C_3

Proof: Let $G = C_3$ or $K_{1,q}$ then it is easily verified that $m_{ee}(G) = q$. Conversely, let $m_{ee}(G) = q$. Suppose $G \neq C_3$ and $G \neq K_{1,q}$, Then by Theorem: 2.13 $m_{ee}(G) \leq q - 1$ which is a contradiction. Therefore $m_{ee}(G) = q$.■

The following theorem gives a realization result.

Theorem: 2.15. For each pair of integers k and q with $2 \leq k \leq q$, there exists a connected graph G of order $q + 1$ and size q with $m_{ee}(G) = k$.

Proof. For $2 \leq k \leq q$, let P be a path of order $q - k + 3$. Let G be the graph obtained from P by adding $k - 2$ new vertices to P and joining them to any cut-vertex of P. Clearly, G is a tree of order $q + 1$ and size q with k end-edges and so by Theorem:2.10, $m_{ee}(G) = k$. ■

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