Convex topological Space and some Continuous functions

Bijoy Samanta
Assistant Professor in Mathematics, Department of Mathematics , Shibpur Dinobundhoo Institution (College), 412/1 , G.T. Road (S) , Shibpur , Howrah – 711 102 , W.B. , India.

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ABSTRACT *Some continuous functions has been introduced using both the topology τ and convexity $\mathcal{C}$ on the same underlying set X where $(X, \tau, \mathcal{C})$ is termed as convex topological space and inter relation with examples among them are also investigated .*

Keywords: *Convex topological space , $\tau - \mathcal{C}$ semi compatible , $\delta - \mathcal{C}$ continuous function , $\theta - \mathcal{C}$ continuous function , almost $\mathcal{C}$ continuous function .*
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1. Introduction : The development of "abstract convexity" has emanated from different sources in different ways ; the first type of development basically banked on generalization of particular problems such as separation of convex sets [3] , extremality [4] , [2] or continuous selection [12] . The second type of development lay before the reader such axiomatizations , which in every case of design , express particular point of view of convexity . With the view point of generalized topology which enters into convexity via the closure or hull operator , Schmidt [1953] and Hammer [1955] , [1963] , [1963b] introduced some axioms to explain abstract convexity . The arising of convexity from algebraic operations and the related property of domain finiteness receive attentions in Birchoff and Frink [1948], Schmidt [1953] , Hammer [1963] .

In [15] the author has discussed " Topology and Convexity on the same set" and introduced the compatibility of the topology with a convexity on the same underlying set . At the very early stage of this paper we have set aside this concept of compatibility and started just with a triplet $(X, \tau, \mathcal{C})$ and call it convex topological space only to bring back "compatibility" in another way subsequently . With this compatibility , Van De Val has called the triplet $(X, \tau, \mathcal{C})$ a topological convex structure .

In this paper , Art. 2 deals with some early definitions , results and in Art. 3 we have discussed mainly inter relation among different types of continuous functions .

2. Prerequisites :

Definition 2.1 : [15] Let X be a non empty set . A family $\mathcal{C}$ of subsets of the set X is called a convexity on X if

1. $\phi, X \in \mathcal{C}$
2. $\mathcal{C}$ is stable for intersection , i.e. if $\mathcal{D} \subseteq \mathcal{C}$ is non empty , then $\cap \mathcal{D} \in \mathcal{C}$
3. $\mathcal{C}$ is stable for nested unions , i.e. if $\mathcal{D} \subseteq \mathcal{C}$ is non empty and totally ordered by set inclusion , then $\cup \mathcal{D} \in \mathcal{C}$.

The pair $(X, \mathcal{C})$ is called a convex structure . The members of $\mathcal{C}$ are called convex sets and their complements are called concave sets .

Definition 2.2 : [15] Let $\mathcal{C}$ be a convexity on set X . Let $A \subseteq X$. The convex hull of A is denoted by $co(A)$ and defined by $co(A) = \cap \{C : A \subseteq C \in \mathcal{C}\}$.

Note 2.3 : [15] Let $(X, \mathcal{C})$ be a convex structure and let Y be a subset of X . The family of sets $\mathcal{C}_Y = \{C \cap Y : C \in \mathcal{C}\}$ is a convexity on Y; called the relative convexity of Y.

Note 2.4 : [15] The hull operator co_Y of a subspace $(Y, \mathcal{C}_Y)$ satisfy the following :

$$\forall A \subseteq Y : co_Y(A) = co(A) \cap Y .$$

Definition 2.5 : [5] Let (X, τ) be a topological space and let $\mathcal{C}$ be a convexity on X . Then the triplet $(X, \tau, \mathcal{C})$ is called a convex topological space (CTS in short) .

Theorem 2.6 : [5] Let $(X, \tau, \mathcal{C})$ be a convex topological space . Let A be a subset of X . Consider the set A_* , where A_* is defined as follows : $A_* = \{x \in X : co(U) \cap A \neq \emptyset, x \in U \in \tau\}$. Then the collection $\tau_* = \{A^c : A \subseteq X, A = A_*\}$ is a topology on X such that $\tau_* \subseteq \tau$.

Note 2.7 : [5] In a convex topological space $(X, \tau, \mathcal{C})$ a subset A of X is said to be τ_*-closed if $A = A_*$.

Definition 2.8 : [5] Let $(X, \tau, \mathcal{C})$ be a convex topological space . The space $(X, \tau, \mathcal{C})$ is called $\tau - \mathcal{C}$ semi compatible if for every $A \in \tau$, A_* is a τ_*-closed set , i.e. if $A \in \tau$, then $(A_*)_* = A_*$.

Definition 2.9 : [5] Let $(X , \tau , \mathcal{C})$ be a convex topological space and A be a subset of X. Then A is said to be an $R-\mathcal{C}$ open set if $int(A_*) = A$.
A subset B is called an $R-\mathcal{C}$ closed if B^c is an $R-\mathcal{C}$ open set.

Definition 2.10 : [7] Let $(X , \tau , \mathcal{C})$ be a convex topological space. Let S be a subset of X and $x \in X$.

(a) x is called $\delta-\mathcal{C}$ cluster point of S if $S \cap int(U_*) \neq \emptyset$, for each open nbd. U of x.
(b) The family of all $\delta-\mathcal{C}$ cluster points of S is called the $\delta-\mathcal{C}$ closure of S and is denoted by $[S]_{\delta-\mathcal{C}}$.
(c) A subset P of X is called $\delta-\mathcal{C}$ closed if $[P]_{\delta-\mathcal{C}} = P$.

The complement of a $\delta-\mathcal{C}$ closed set is said to be a $\delta-\mathcal{C}$ open set.

Theorem 2.11 : [7] Let $(X , \tau , \mathcal{C})$ be a convex topological space which is $\tau-\mathcal{C}$ semi compatible. Then we have the following properties :

(1) If A is an open set, then $int(A_*)$ is an $R-\mathcal{C}$ open set.
(2) If A and B are $R-\mathcal{C}$ open sets, then so is $A \cap B$.
(3) If A is an $R-\mathcal{C}$ open set, then A is a regular open set.
(4) $A \subseteq [A]_{\delta-\mathcal{C}}$.
(5) If A is an $R-\mathcal{C}$ open set, then it is a $\delta-\mathcal{C}$ open set.
(6) Every $\delta-\mathcal{C}$ open set is the union of family of $R-\mathcal{C}$ open sets.

Theorem 2.12 : [7] Let A and B be subsets of a convex topological space $(X , \tau , \mathcal{C})$ which is $\tau-\mathcal{C}$ semi compatible. Then the following properties hold :

(1) $A \subseteq B \Rightarrow [A]_{\delta-\mathcal{C}} \subseteq [B]_{\delta-\mathcal{C}}$.
(2) $[A]_{\delta-\mathcal{C}} = \bigcap\{F \subseteq X : A \subseteq F \text{ and } F \text{ is } \delta-\mathcal{C} \text{ closed}\}$.
(3) If A_α is $\delta-\mathcal{C}$ closed sets of X for each $\alpha \in \Lambda$, then so is $\bigcap_{\alpha\in\Lambda}(A_\alpha)$.
(4) $[A]_{\delta-\mathcal{C}}$ is $\delta-\mathcal{C}$ closed set.

Remark 2.13 : [7] $[A]_{\delta-\mathcal{C}}$ is the smallest $\delta-\mathcal{C}$ closed set containing A.

Theorem 2.14 : [7] Let $(X , \tau , \mathcal{C})$ be a convex topological space which is $\tau-\mathcal{C}$ semi compatible. Let $\tau_{\delta-\mathcal{C}} = \{A \subseteq X : A \text{ is a } \delta-\mathcal{C} \text{ open set in } X\}$. Then $\tau_{\delta-\mathcal{C}}$ is a topology on X such that $\tau_{\delta-\mathcal{C}} \subseteq \tau$.

Definition 2.15 : [8] Let $(X , \tau , \mathcal{C}_1)$ and $(Y , \sigma , \mathcal{C}_2)$ be two convex topological spaces. A function $f : (X , \tau , \mathcal{C}_1) \rightarrow (Y , \sigma , \mathcal{C}_2)$ is said to be $\delta-\mathcal{C}$ continuous if for each $x \in X$ and each open nbd. V of $f(x)$, there exists an open nbd. U of x such that $f\big(int(U_*)\big) \subseteq int(V_*)$.

Theorem 2.16 : [8] Let $(X , \tau , \mathcal{C}_1)$ and $(Y , \sigma , \mathcal{C}_2)$ be two convex topological spaces where $(X , \tau , \mathcal{C}_1)$ is $\tau-\mathcal{C}_1$ semi compatible and $(Y , \sigma , \mathcal{C}_2)$ is $\sigma-\mathcal{C}_2$ semi compatible. For a function $f : (X , \tau , \mathcal{C}_1) \rightarrow (Y , \sigma , \mathcal{C}_2)$ the following are equivalent :

1. f is $\delta-\mathcal{C}$ continuous.
2. For each $x \in X$ and each $R-\mathcal{C}$ open set V containing $f(x)$, there exists an $R-\mathcal{C}$ open set containing x such that $f(U) \subseteq V$.
3. $f([A]_{\delta-\mathcal{C}}) \subseteq [f(A)]_{\delta-\mathcal{C}}$, for each $A \subseteq X$.
4. $[f^{-1}(B)]_{\delta-\mathcal{C}} \subseteq f^{-1}([B]_{\delta-\mathcal{C}})$, for every $B \subseteq Y$.
5. For every $\delta-\mathcal{C}$ closed set F of Y, $f^{-1}(F)$ is $\delta-\mathcal{C}$ closed set in X.
6. For every $\delta-\mathcal{C}$ open set V of Y, $f^{-1}(V)$ is $\delta-\mathcal{C}$ open set in X.
7. For every $R-\mathcal{C}$ open set V of Y, $f^{-1}(V)$ is $\delta-\mathcal{C}$ open set in X.
8. For every $R-\mathcal{C}$ closed set F of Y, $f^{-1}(F)$ is $\delta-\mathcal{C}$ closed set in X.

Corollary 2.17 : [8] Let $(X , \tau , \mathcal{C}_1)$ and $(Y , \sigma , \mathcal{C}_2)$ be two convex topological spaces where $(X , \tau , \mathcal{C}_1)$ is $\tau-\mathcal{C}_1$ semi compatible and $(Y , \sigma , \mathcal{C}_2)$ is $\sigma-\mathcal{C}_2$ semi compatible. A function $f : (X , \tau , \mathcal{C}_1) \rightarrow (Y , \sigma , \mathcal{C}_2)$ is $\delta-\mathcal{C}$ continuous iff $f: \big(X , \tau_{\delta-\mathcal{C}_1}\big) \rightarrow (Y , \sigma_{\delta-\mathcal{C}_2})$ is continuous.

3. Comparison of different types of continuous functions :

Definition 3.1 : Let $(X , \tau , \mathcal{C}_1)$ and $(Y , \sigma , \mathcal{C}_2)$ be two convex topological spaces. A function $f : (X , \tau , \mathcal{C}_1) \rightarrow (Y , \sigma , \mathcal{C}_2)$ is said to be strongly $\theta-\mathcal{C}$ continuous [respectively $\theta-\mathcal{C}$ continuous, almost $\mathcal{C}$ continuous] if for each $x \in X$ and each open nbd. V of $f(x)$, there exists an open nbd. U of x such that $f(U_*) \subseteq V$ [respectively $f(U_*) \subseteq V_*$, $f(U) \subseteq int(V_*)$].

Theorem 3.2 : (a) If a function $f : (X , \tau , \mathcal{C}_1) \rightarrow (Y , \sigma , \mathcal{C}_2)$ is strongly $\theta-\mathcal{C}$ continuous and $g : (Y , \sigma , \mathcal{C}_2) \rightarrow (Z , \gamma, \mathcal{C}_3)$ is almost $\mathcal{C}$ continuous, then $g \circ f : (X , \tau , \mathcal{C}_1) \rightarrow (Z , \gamma, \mathcal{C}_3)$ is $\delta-\mathcal{C}$ continuous.

(b) The following implications hold :
strongly $\theta-\mathcal{C}$ continuous $\Rightarrow \delta-\mathcal{C}$ continuous $\Rightarrow$ almost $\mathcal{C}$ continuous .

Proof : (a) Let $x \in X$ and W be any open set containing $(g \circ f)(x)$. Since g is almost $\mathcal{C}$ continuous , there exists an open nbd. V of $f(x)$ in Y such that $g(V) \subseteq int(W_*)$. Again since f is strongly $\theta-\mathcal{C}$ continuous , there exists an open nbd. U of x in X such that $f(U_*) \subseteq V$. So we get $g(f(U_*)) \subseteq g(V)$. Now $g\left(f\left(int(U_*)\right)\right) \subseteq g(f(U_*)) \subseteq g(V) \subseteq int(W_*) \Rightarrow (g \circ f)\left(int(U_*)\right) \subseteq int(W_*)$. Hence $g \circ f$ is $\delta-\mathcal{C}$ continuous.

(b) Let f be strongly $\theta-\mathcal{C}$ continuous . Also let $x \in X$ and V be any open nbd. of $f(x)$. Then there exists an open nbd. U of x in X such that $f(U_*) \subseteq V$. Now $f\left(int(U_*)\right) \subseteq f(U_*) \subseteq V = int(V) \subseteq int(V_*)$. Hence f is $\delta-\mathcal{C}$ continuous .

Again let f be $\delta-\mathcal{C}$ continuous . Also let $x \in X$ and V be any open nbd. of $f(x)$ in Y. Then there exists an open nbd. U of x in X such that $f\left(int(U_*)\right) \subseteq int(V_*)$. Now $U = int(U) \subseteq int(U_*) \Rightarrow f(U) \subseteq f\left(int(U_*)\right) \subseteq int(V_*)$. Thus f is almost $\mathcal{C}$ continuous .

Remark 3.3 : The following examples show that none of these implications in the above Theorem 3.2 is reversible .

Example 3.4 : Let $X = \{ a , b , c \}$, $\tau = \{ \phi , X , \{ a \} \}$, $\mathcal{C}_1 = \{ \phi , X \}$, $\sigma = \{ \phi , X , \{ b \} \}$, $\mathcal{C}_2 = \{ \phi , X \}$ and the function $f : (X , \tau , \mathcal{C}_1) \to (X , \sigma , \mathcal{C}_2)$ be the identity function on X i.e. , $f = I_X$. Here f is $\delta-\mathcal{C}$ continuous but not strongly $\theta-\mathcal{C}$ continuous .

Since for any $V \in \sigma$, we have $V_* = X$, we conclude that f is $\delta-\mathcal{C}$ continuous . Consider the point b in$(X , \tau , \mathcal{C}_1)$. Now $\{ b \}$ is an open nbd. of $b = f(b)$ in $(X , \sigma , \mathcal{C}_2)$. But there is no open nbd. U of b in $(X , \tau , \mathcal{C}_1)$ such that $f(U_*) \subseteq \{ b \}$. Hence f is not strongly $\theta-\mathcal{C}$ continuous .

Example 3.5 : Let $X = \{ a , b , c \}$, $\tau = \{ \phi , X , \{ a \} \}$, $\mathcal{C}_1 = \{ \phi , X \}$, $\sigma = \{ \phi , X , \{ a \} , \{ b \} , \{ a , b \} \}$, $\mathcal{C}_2 = \{ \phi , X , \{ b \} \}$, and the function $f : (X , \tau , \mathcal{C}_1) \to (X , \sigma , \mathcal{C}_2)$ be the identity function on X . Here f is almost $\mathcal{C}$ continuous but not $\delta-\mathcal{C}$ continuous function .

Clearly f is almost $\mathcal{C}$ continuous function . Now consider the point a in $(X , \tau , \mathcal{C}_1)$ and $V = \{ a \}$ be an open nbd. of $a = f(a)$ in $(X , \sigma , \mathcal{C}_2)$. In $(X , \sigma , \mathcal{C}_2)$, we have $int(V_*) = int(\{a\}_*) = int(\{ a , c \}) = \{ a \}$. Again in $(X , \tau , \mathcal{C}_1)$ open nbd. of $\{ a \}$ are $\{ a \}$ and X and $int(\{a\}_*) = int(X) = X$, $int(X_*) = X$. Thus there is no open nbd. of U of a in $(X , \tau , \mathcal{C}_1)$ such that $f\left(int(U_*)\right) \subseteq int(V_*)$. Hence f is not $\delta-\mathcal{C}$ continuous function .

Definition 3.6 : A convex topological space $(X , \tau , \mathcal{C})$ is said to be an $S\mathcal{C}-R$ space if for each $x \in X$ and each open nbd. V of x there exists an open nbd. U of x such that $x \in U \subseteq int(U_*) \subseteq V$.

Theorem 3.7 : For a function $f : (X , \tau , \mathcal{C}_1) \to (Y , \sigma , \mathcal{C}_2)$ the following properties are true :

(a) If Y is an $S\mathcal{C}-R$ space and f is $\delta-\mathcal{C}$ continuous , then f is continuous .

(b) If X is an $S\mathcal{C}-R$ space and f is almost $\mathcal{C}$ continuous , then f is $\delta-\mathcal{C}$ continuous .

Proof : (a) Let Y be an $S\mathcal{C}-R$ space and $x \in X$. Then for each open nbd. V of $f(x)$, there exists an open nbd. W of $f(x)$ such that $f(x) \in W \subseteq int(W_*) \subseteq V$. Since f is $\delta-\mathcal{C}$ continuous , there exists an open nbd. U of x such that $f\left(int(U_*)\right) \subseteq int((W_*)$. Since U is an open set , $f(U) = f\left(int(U)\right) \subseteq f\left(int(U_*)\right) \subseteq int(W_*) \subseteq V$ $i.e.,$ $f(U) \subseteq V$. Hence f is continuous .

(b) Let $x \in X$ and V be an open nbd. of $f(x)$. Since f is almost $\mathcal{C}$ continuous , there exists an open nbd. U of x such that $f(U) \subseteq int(V_*)$. Again since X is an $S\mathcal{C}-R$ space there exists an open nbd. W of x such that $int(W_*) \subseteq U$. Thus $f\left(int(W_*)\right) \subseteq f(U) \subseteq int(V_*)$. Hence f is $\delta-\mathcal{C}$ continuous .

Corollary 3.8 : If $(X , \tau , \mathcal{C}_1)$ and $(Y , \sigma , \mathcal{C}_2)$ are $S\mathcal{C}-R$ spaces , then the concepts on a function $f : (X , \tau , \mathcal{C}_1) \to (Y , \sigma , \mathcal{C}_2)$, $\delta-\mathcal{C}$ continuity , continuity , almost $\mathcal{C}$ continuity are equivalent .

Definition 3.9 : A function $f : (X , \tau , \mathcal{C}_1) \to (Y , \sigma , \mathcal{C}_2)$ is said to be almost $\mathcal{C}-open$ if for each $R-\mathcal{C}$ open set U in X , $f(U)$ is open in Y .

Theorem 3.10 : If a function $f : (X , \tau , \mathcal{C}_1) \to (Y , \sigma , \mathcal{C}_2)$ is $\theta-\mathcal{C}$ continuous and almost $\mathcal{C}-open$ where $(X , \tau , \mathcal{C}_1)$ is $\tau-\mathcal{C}_1$ semi compatible , then f is $\delta-\mathcal{C}$ continuous .

Proof : Let $x \in X$ and V be an open nbd. of $f(x)$. Since f is $\theta-\mathcal{C}$ continuous , there exists an open nbd. U of x such that $f(U_*) \subseteq V_*$. Thus $f\left(int(U_*)\right) \subseteq f(U_*) \subseteq V_*$. Now $int(U_*)$ is an $R-\mathcal{C}$ open set in X . Since f is almost $\mathcal{C}-open$, $f\left(int(U_*)\right)$ is an open set in Y which is contained in V_* . So we have , $f\left(int(U_*)\right) \subseteq int(V_*)$. Hence f is a $\delta-\mathcal{C}$ continuous function .

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