A TRACE INEQUALITY FOR POSITIVE DEFINITE QUATERNION HERMITIAN MATRICES

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ABSTRACT *In this note we prove that Tr{MN+PQ} > 0 when the following two conditions are met:*
(i) The quaternion hermitian matrices M,N,P,Q are structured as follows M= A-B, $N = B^{-1} - A^{-1}$, *P = C - D, Q =* $(B+D)^{-1} + (A+C)^{-1}$
(ii)A,B are positive definite quaternion hermitian matrices and C,D are positive semidefinite quaternian hermitian matrices.

Keywords: *Positive semi definite, Positive definite, Hermitian matrices, Quaternion matrices.*

1. Introduction :

Trace inequalities are useful in many applications. For example, trace inequalities naturally arise in control theory (see e.g., [9]) and in communication system with multiple input and multiple output (see eg., [11]). In this paper, the authors prove an inequality for which one application has already been identified : the uniqueness of a pure Nash equilibrium in concave games. Indeed, the reader will be able to check that the proposed inequality allows one to generalize the diagonally strict concavity condition introduce by Rosen in [10] to concave communication games with matrix strategies [1].

Let us start with scalar case. Let α,β,γ,δ be four reals such that α > 0, β > 0, $\gamma \geq 0$, δ ≥ 0. Then it can be checked that we have the following inequality.

$(\alpha-\beta)\left(\frac{1}{\beta}-\frac{1}{\alpha}\right) + (\gamma-\delta)\left(\frac{1}{\beta+\delta}-\frac{1}{\alpha+\gamma}\right) \geq 0$. The main issue addressed here is to show that this inequality has a quaternion hermitian counterparts. i.e., we want to prove the following Lemmas and Theorem.

2. Auxiliary Results and Theorem:

The following Lemmas and Theorem is easy to prove with the help of quaternion hermitian matrices.

Lemma 2.1 :

Let A,B be two positive definite quaternian hermitian matrices. Then

$$T = Tr\{(A-B)*(B^{-1}-A^{-1})\} \geq 0 \qquad \text{..... (2.1)}$$

Proof:

Let, $A = A_0 + A_1 j + A_2 k$, $B = B_0 + B_1 j + B_2 k$,
$A^{-1} = A_0^{-1} + A_1^{-1} j + A_2^{-1} k$, $B^{-1} = B_0^{-1} + B_1^{-1} j + B_2^{-1} k$

here,

$$A - B = A_0 + A_1 j + A_2 k - (B_0 + B_1 j + B_2 k) = (A_0 - B_0) + (A_1 - B_1)j + (A_2 - B_2)k$$

$$B^{-1} - A^{-1} = B_0^{-1} + B_1^{-1} j + B_2^{-1} k - (A_0^{-1} + A_1^{-1} j + A_2^{-1} k)$$
$$= (B_0^{-1} - A_0^{-1}) + (B_1^{-1} - A_1^{-1})j + (B_2^{-1} - A_2^{-1})k$$

$$T = \{(A-B)*(B^{-1} - A^{-1})\} = \{(A_0 - B_0) + (A_1 - B_1)j + (A_2 - B_2)k\} * \{(B_0^{-1} - A_0^{-1}) + (B_1^{-1} - A_1^{-1})j + (B_2^{-1} - A_2^{-1})k\}$$

$$\mathrm{T} = \{(A_0 - B_0)(B_0^{-1} - A_0^{-1}) + (A_1 - B_1)(B_1^{-1} - A_1^{-1})j + (A_2 - B_2)(B_2^{-1} - A_2^{-1})k\} \geq 0$$

$$\mathrm{T} = \{(A-B)*(B^{-1} - A^{-1}) \geq 0\}$$

[since $A * B = A_0 B_0 + A_1 B_1 j + A_2 B_2 k$ and A,B are positive definite quaternion hermitian matrices]
That is,

$$\mathrm{T}=\{(A-B)*(B^{-1} - A^{-1})\} \geq 0$$

Then,

$$\mathrm{T}=Tr\,\{(A-B)*(B^{-1} - A^{-1})\} \geq Tr\,(A-B) * Tr(B^{-1} - A^{-1})$$
$$\mathrm{T}=Tr\,\{(A-B) = Tr[(A_0 - B_0) + (A_1 - B_1)j + (A_2 - B_2)k]$$

$T = Tr\ (B^{-1} - A^{-1}) = Tr\ [(B_0^{-1} - A_0^{-1}) + (B_1^{-1} - A_1^{-1})j + (B_2^{-1} - A_2^{-1})k]$

So, T= $Tr\{(A - B) * (B^{-1} - A^{-1})\} = Tr\{[(A_0 - B_0) + (A_1 - B_1)j + (A_2 - B_2)k] * [(B_0^{-1} - A_0^{-1}) + (B_1^{-1} - A_1^{-1})j + (B_2^{-1} - A_2^{-1})k]\} \geq 0$

[since $A * B = A_0B_0 + A_1B_1j + A_2B_2k$ and A,B are positive definite quaternion hermtian matrices]

$T = Tr\{\ (A - B) * (B^{-1} - A^{-1})\} \geq Tr(A - B) * Tr(B^{-1} - A^{-1}) \geq 0$

Therefore,T= $Tr\ \{(A - B) * (B^{-1} - A^{-1})\} \geq 0$

The proof is complete.

Lemma 2.2

Let, M and N be two positive semi definite quaternion hermitian matrices.

Then Tr{M*N} ≥ 0 ..... (2.2)

Proof :

Let, M and N be two positive semi definite quaternion hermitian matrices.

So, $M = M_0 + M_1j + M_2k$

$N = N_0 + N_1j + N_2k$

$M * N = M_0N_0 + M_1N_1j + M_2N_2k$ [since $A * B = A_0B_0 + A_1B_1j + A_2B_2k$]

$Tr\{M * N\} \geq Tr(M) * Tr(N) \geq 0$ [Since M and N are positive semi definite quaternion hermitian -matrices]

Now,

$$Tr\{M * N\} \geq Tr\ (M_0 + M_1j + M_2k) * Tr(N_0 + N_1j + N_2k) \geq 0 \geq Tr(M) * Tr(N) \geq 0$$
$$= Tr\ (M_0N_0 + M_1N_1j + M_2N_2k)$$
$$= Tr(M_0N_0) + Tr(M_1N_1)j + Tr(M_2N_2)k$$
$$\geq Tr(M_0)Tr(N_0) + Tr(M_1)Tr(N_1)j + Tr(M_2)Tr(N_2)k$$

Since $Tr(M_S) * Tr(N_s) \geq 0\ where,\ s = 0,1,2$

This implies that Tr{M*N} ≥ 0

The proof is completed.

Lemma 2.3

Let A,B be two positive definite quaternion hermitian matrices and C,D be two positive semi definite quaternion hermition matrices whereas X is only assumed to be quaternion hermitian.

Then $Tr\{XA^{-1} * XB^{-1}\} - Tr\{X\ (A + C)^{-1} * X(B + D)^{-1}\} \geq 0$ ..... (2.3)

Proof:

Let, $A = A_{0+}A_1j + A_2K,\ B = B_0 + B_1j + B_2k,\ X = X_0 + X_1j + X_2k,$

$C = C_0 + C_1j + C_2k,\ D = D_0 + D_1j + D_2k,$

$A^{-1} = A_0^{-1} + A_1^{-1}j + A_2^{-1}k,\ B^{-1} = B_0^{-1} + B_1^{-1}j + B_2^{-1}k$

$XA^{-1} = X_0A_0^{-1} + X_1A_1^{-1}j + X_2A_2^{-1}k$

$XB^{-1} = X_0B_0^{-1} + X_1B_1^{-1}j + X_2B_2^{-1}k$

$XA^{-1} * XB^{-1} = (X_0A_0^{-1} + X_0B_0^{-1}) + (X_1A_1^{-1} + X_1B_1^{-1})j + (X_2A_2^{-1} + X_2B_2^{-1})k$

$Tr\{XA^{-1} * XB^{-1}\} \geq Tr(X_0A_0^{-1} + X_0B_0^{-1}) + Tr(X_1A_1^{-1} + X_1B_1^{-1})j + Tr(X_2A_2^{-1} + X_2B_2^{-1})k \geq 0$

So, $Tr\{XA^{-1} * XB^{-1}\} \geq 0$ [since $A * B = A_0B_0 + A_1B_1j + A_2B_2k$] ..... (2.4)

$(A + C)^{-1} = (A_0 + C_0)^{-1} + (A_1 + C_1)^{-1}j + (A_2 + B_2)^{-1}k$

$(B + D)^{-1} = (B_0 + D_0)^{-1} + (B_1 + D_1)^{-1}j + (B_2 + D_2)^{-1}k$

$$X(A + C)^{-1} * X(B + D)^{-1} = (X_0(A_0 + C_0)^{-1} + X_1(A_1 + C_1)^{-1}j + X_2(A_2 + C_2)^{-1}k) * (X_0(B_0 + D_0)^{-1} + X_1(B_1 + C_1)^{-1}j + X_2(B_2 + C_2)^{-1}k)$$

$$Tr\{X(A + C)^{-1} * X(B + D)^{-1}\} = Tr\ \{\ [X_0(A_0 + C_0)^{-1} + X_1(A_1 + C_1)^{-1}j + X_2(A_2 + C_2)^{-1}k] * [X_0(B_0 + D_0)^{-1} + X_1(B_1 + C_1)^{-1}j + X_2(B_2 + C_2)^{-1}k]\} \geq 0$$

By Lemma 2.2, $Tr(M * N) \geq Tr(M) * Tr(N)$

Therefore, $Tr\{X(A + C)^{-1} * X(B + D)^{-1}\} \geq Tr[X(A + C)^{-1}] * Tr[X(B + D)^{-1}] \geq 0$

So, $Tr\{X(A + C)^{-1} * X(B + D)^{-1}\} \geq 0$ ..... (2.5)

From (2.4) – (2.5), we get $Tr\{XA^{-1} * XB^{-1}\} - Tr\{X(A+C)^{-1} * X(B+D)^{-1}\} \geq 0$
The proof is completed .

Theorem: 2.4

Let A,B be two positive definite quaternion hermitian matrices and C,D , two positive semi definite quaternian hermitian matrices. Then

$$T = Tr\{(A-B) * (B^{-1} - A^{-1}) + (C-D) * [(B+D)^{-1} - (A+C)^{-1}]\} \geq 0.$$

Proof:

Let, $A = A_0 + A_1 j + A_2 k,\ B = B_0 + B_1 j + B_2 k,$
$A^{-1} = A_0^{-1} + A_1^{-1} j + A_2^{-1} k,\ B^{-1} = B_0^{-1} + B_1^{-1} j + B_2^{-1} k$

here,
$A - B = A_0 + A_1 j + A_2 k - (B_0 + B_1 j + B_2 k) = (A_0 - B_0) + (A_1 - B_1) j + (A_2 - B_2) k$
$B^{-1} - A^{-1} = B_0^{-1} + B_1^{-1} j + B_2^{-1} k - (A_0^{-1} + A_1^{-1} j + A_2^{-1} k)$
$= (B_0^{-1} - A_0^{-1}) + (B_1^{-1} - A_1^{-1}) j + (B_2^{-1} - A_2^{-1}) k$
$T = \{(A-B) * (B^{-1} - A^{-1})\} = \{(A_0 - B_0) + (A_1 - B_1) j + (A_2 - B_2) k\} * \{(B_0^{-1} - A_0^{-1}) + (B_1^{-1} - A_1^{-1}) j + (B_2^{-1} - A_2^{-1}) k\}$
T = $\{(A_0 - B_0)(B_0^{-1} - A_0^{-1}) + (A_1 - B_1)(B_1^{-1} - A_1^{-1}) j + (A_2 - B_2)(B_2^{-1} - A_2^{-1}) k\} \geq 0$
T = $\{(A-B) * (B^{-1} - A^{-1}) \geq 0\}$
[since $A * B = A_0 B_0 + A_1 B_1 j + A_2 B_2 k$ and A,B are positive definite quaternion hermitian matrices]
That is, T=$\{(A-B) * (B^{-1} - A^{-1})\} \geq 0$
Then,
T=$Tr\ \{(A-B) * (B^{-1} - A^{-1})\} \geq Tr\ (A-B) * Tr(B^{-1} - A^{-1})$
T=$Tr\ \{(A-B) = Tr[(A_0 - B_0) + (A_1 - B_1) j + (A_2 - B_2) k]$
$T = Tr\ (B^{-1} - A^{-1}) = Tr\ [(B_0^{-1} - A_0^{-1}) + (B_1^{-1} - A_1^{-1}) j + (B_2^{-1} - A_2^{-1}) k]$
So, T= $Tr\{(A-B) * (B^{-1} - A^{-1})\} = Tr\{[(A_0 - B_0) + (A_1 - B_1) j + (A_2 - B_2) k] * [(B_0^{-1} - A_0^{-1}) + (B_1^{-1} - A_1^{-1}) j + (B_2^{-1} - A_2^{-1}) k]\} \geq 0$
[since $A * B = A_0 B_0 + A_1 B_1 j + A_2 B_2 k$ and A,B are positive definite quaternion hermtian matrices]
$T = Tr\{(A-B) * (B^{-1} - A^{-1})\} \geq Tr(A-B) * Tr(B^{-1} - A^{-1}) \geq 0$
Therefore, $T = Tr\ \{(A-B) * (B^{-1} - A^{-1})\} \geq 0$ …… (2.5)
Then, Let $C = C_0 + C_1 j + C_2 K,\ D = D_{0+} D_1 j + D_2 k$

Now,$T = (C-D) * [(B+D)^{-1} - (A+C)^{-1}] = \big((C_0 - D_0) + (C_1 - D_1) j + (C_2 - D_2) k\big) * [(B_0 + D_0) + (B_1 + D_1) j + (B_2 + D_2) k)^{-1} - ((A_0 + C_0) + (A_1 + C_1) j + (A_2 + C_2) k)^{-1}$

$T = Tr\{(C-D) * [(B+D)^{-1} - (A+C)^{-1}]\} = Tr\{\big((C_0 - D_0) + (C_1 - D_1) j + (C_2 - D_2) k\big) * [(B_0 + D_0) + (B_1 + D_1) j + (B_2 + D_2) k)^{-1} - ((A_0 + C_0) + (A_1 + C_1) j + (A_2 + C_2) k)^{-1}]\}$

$\geq Tr\{\big((C_0 - D_0) + (C_1 - D_1) j + (C_2 - D_2) k\big) * Tr[(B_0 + D_0) + (B_1 + D_1) j + (B_2 + D_2) k)^{-1} - ((A_0 + C_0) + (A_1 + C_1) j + (A_2 + C_2) k)^{-1}]\}$

$\geq Tr(C-D) * Tr[(B+D)^{-1} - (A+C)^{-1}] \geq 0$
$\geq Tr\{(C-D) * [(B+D)^{-1} - (A+C)^{-1}]\} \geq 0$
Therefore, $T = Tr\{(C-D) * [(B+D)^{-1} - (A+C)^{-1}]\} \geq 0$ ……(2.6)
Adding (2.5) and (2.6), we have
$T = Tr\ \{(A-B) * (B^{-1} - A^{-1})\} + Tr\{(C-D) * [(B+D)^{-1} - (A+C)^{-1}]\} \geq 0$
That is $T = Tr\ \{(A-B) * (B^{-1} - A^{-1}) + (C-D) * [(B+D)^{-1} - (A+C)^{-1}]\} \geq 0$
The proof is complete

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