QUEUEING ANALYSIS OF M/M/1 QUEUEING SYSTEM WITH DETAINMENT OF RETRACTED CUSTOMERS IN STATE DEPENDENT ARRIVAL PROCESS

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ABSTRACT *Customer satisfaction plays an important role in different sales and services industries. If customers are not satisfied with the service quality and way of service from service provider, then it plays negative impact on growth of the company and customers can be reneged from service. Reneging of customers from the service of the company again play negative impact on business of the company. The reneging customers can be retained for further service in the queueing system with some probability by giving special offers and additional benefits to customers. In this research paper, we have proposed single server queueing model having state dependent arrivals and retention of reneged customers. The solution of the Markovian queueing model has been derived by iterative procedure and some important measures of performance have been derived for the queueing model developed with some special cases.*

Keywords: *Arrival process, retracted customers, Markov queueing model, reneging, Poisson process.*

I.INTRODUCTION

Queueing theory plays key role in modeling of real life problems involving congestions in vide areas of service industries like bus stand, railway station, car service station, railway checking points, toll tax plaza etc. Queueing application with impatience can be seen in traffic modeling, business, sell and service and medical industries etc. In real life situations, the server may warm up when seeing a long queue forming in service facility. Much time the server becomes flustered and the mean service rate also decreases. In actual life, many queueing situations may arise during the service of customers due to which customers may discouraged due to a long queue forming. So, the customer can decide not to join the queue i.e. the customer may balk or may depart after joining the queue without getting served due to lack of patience. The important role of the system appears in many real life problems such as the situations involving impatient cellular customers and emergency rooms handling for critical patients etc. Queueing systems with reneging, balking or both were studied by many researchers. Reneging phenomenon of single channel queue analyzed by Obert [1]. Haight[2] first proposed the M/M/1 queueing model with balking behavior of customers. Al-Seedy and Kotb [3] developed the transient solution of a single server system with balking behavior of customers. The M/M/1 single server queueing model with customers reneging behavior was also proposed by [4].

The effects of balking and reneging behaviors on some queueing models analyzed by Ancker and Gafarian [5]. Some queueing problems discussed by Ancker and Gafarian [6]. M/M/1/N queueing model with server vacation and balking, reneging behavior of customers was discussed by Zhang et al. [7]. Al-Seedy et al. [8] proposed M/M/c multi server queueing model with balking and reneging behavior of customers and solution was obtained using generating function technique. Jain et al. [9] discussed unreliable M/M/2/K queueing finite capacity queueing model under N and F-policy with multi optional phase repair. The strategic behaviors of customers were examined by Wang et al. [10] in an M/M/1 constant retrial queue with the N-policy. Hadidi [11] carried out analysis of busy period processes for M/Mn/1 and Mn/M/1 single server queuing models with state dependent service and arrival rates. He also discussed results for busy period and transient state probabilities. Wang and Yang [12] analyzed the steady-state solutions for the finite capacity Monrovian queue having single unreliable server in the system. To determine the queue length distribution, they have employed the matrix analytical approach.

The all above discussed queuing models deal with customer impatience of customers and discouragement. Different extensions of customer impatience in single-server and multi-server queues are carried out here. The queueing system will have a negative impact if the customers are not satisfied with the service. Queueing model with customer impatience has vast application in service system, bio-medical modeling, and communication network etc. System is highly affected by customer impatience and can loss their potential customers. We can apply some strategies to retain impatient customers in the system. The reneged customer can be made to stay for further service in the queueing system. Taking all these concepts into consideration, a single server finite capacity queueing model with reneging and retention of reneged customers has been developed in this paper. Rest of the paper is arranged as follows: Formation of queueing model has been described in section 2. In section 3, solution of the queueing model is described. In section 4, Performance measures for the queueing model have been derived. In section 5, some special cases of the queueing model have been discussed. Conclusion of the research paper is described in section 6.

II.FORMULATION OF QUEUEING MODEL

In this section, we formulate the queueing model for the problem. The queueing model investigated in this paper is based on the following assumptions:

(i) A single server queueing system in which the customers arrive in a Poisson fashion with state dependent arrival rate that depends on the number of customers present in the system at that time i.e. $\frac{\lambda}{(k+1)}$.

(ii) The service times are distributed independently, identically and exponentially with parameters μ, μ_1.

(iii) The customers are served in FCFS discipline.

(iv) The capacity of the system is finite K.

(v) When customer joins the queue he waits for service for a certain time. The customer becomes impatient (reneged) if he is not served in that time and may leave the queue with probability r and may remain in the queue for his service with probability $r = 1 - r$.

(vi) υ is a reneging rate of the customer

(vii) λ is mean arrival rate of customers

III.SOLUTION OF THE QUEUEING MODEL

Let $p_n(t)$ be the probability that there are n customers in the system at time t. The differential difference equations are derived by using the general birth-death arguments. These equations are solved iteratively in steady state in order to obtain the steady state solution. There are two service rates slow (μ) and fast (μ_1).

Case I: SLOW SERVICE RATE

The differential difference equations for this case are

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \tag{1}$$

$$\frac{dP_k(t)}{dt} = \left[\left(\frac{\lambda}{k+1}\right) + \mu + (k-1)\upsilon r\right] P_k(t) + (\mu + k\upsilon r) P_{k+1}(t) + \left(\frac{\lambda}{k}\right) P_{k-1}(t), k = 1,2,3,..,K-1 \tag{2}$$

$$\frac{dP_K(t)}{dt} = -[\mu + (K-1)\upsilon r] P_K(t) + \left(\frac{\lambda}{K}\right) P_{K-1}(t) \tag{3}$$

In steady state $\lim it_{t\to\infty} P_k(t) = P_k$ and therefore $\frac{dP_k(t)}{dt} = 0$ as $t \to \infty$ and hence the equations (3.1) to (3.3) converted in to following difference equations

$$0 = -\lambda P_0 + \mu P_1 \tag{4}$$

$$0 = \left[\left(\frac{\lambda}{k+1}\right) + \mu_1 + (k-1)\upsilon r\right] P_k + (\mu_1 + k\upsilon r) P_{k+1}(t) + \left(\frac{\lambda}{k}\right) P,\ k = 1,2,3,..,K-1 \tag{5}$$

$$0 = -[\mu_1 + (K-1)\upsilon r] P_K + \left(\frac{\lambda}{K}\right) P_{K-1} \tag{6}$$

Solving iteratively equations (4.4) to (4.6), we get

$$P_k = \frac{1}{k!} \prod_{n=1}^{k} \frac{\lambda}{[\mu + (n-1)\upsilon r]} P_0,\ 1 \le k \le K \tag{7}$$

$$P_0 = \frac{1}{1 + \sum_{k=1}^{K} \left(\frac{1}{k!} \prod_{n=1}^{k} \frac{\lambda}{\mu + (n-1)\upsilon r} \right)} \tag{8}$$

Case II: FAST SERVICE RATE

The differential difference equations for this case are

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu_1 P_1(t) \tag{9}$$

$$\frac{dP_k(t)}{dt} = \left[\left(\frac{\lambda}{k+1}\right) + \mu_1 + (k-1)\upsilon r\right] P_k(t) + (\mu_1 + k\upsilon r) P_{k+1}(t) + \left(\frac{\lambda}{k}\right) P_{k-1}(t), k = 1,2,3,..,K-1 \tag{10}$$

$$\frac{dP_K(t)}{dt} = -[\mu_1 + (K-1)\upsilon r] P_K(t) + \left(\frac{\lambda}{K}\right) P_{K-1}(t) \tag{11}$$

In steady state $\lim it_{t\to\infty} P_k(t) = P_k$ and therefore $\frac{dP_k(t)}{dt} = 0$ as $t \to \infty$ and hence equations (3.9) to (3.11) converted into following equations

$$0 = -\lambda P_0 + \mu P_1 \tag{12}$$

$$0 = \left[\left(\frac{\lambda}{k+1}\right) + \mu_1 + (k-1)\upsilon r\right] P_k + (\mu + k\upsilon r) P_{k+1}(t) + \left(\frac{\lambda}{k}\right) P,\ k = 1,2,3,..,K-1 \tag{13}$$

$$0 = -[\mu + (K-1)\upsilon r] P_K + \left(\frac{\lambda}{K}\right) P_{K-1} \tag{14}$$

Solving iteratively equations (4.12) to (4.14), we get

$$P_k = \frac{1}{k!} \prod_{n=1}^{k} \frac{\lambda}{[\mu_1 + (n-1)\upsilon r]} P_0,\ 1 \le k \le K \tag{15}$$

$$P_0 = \frac{1}{1 + \sum_{k=1}^{K} \left(\frac{1}{k!} \prod_{n=1}^{k} \frac{\lambda}{\mu_1 + (n-1)\upsilon r} \right)} \tag{16}$$

IV.PERFORMANCE MEASURES OF QUEUEING MODEL

In this section some performance measures of queuing model are derived. These are useful to study the performance of the queueing system under consideration.

1.1 EXPECTED QUEUE LENGTH(L_q)

Expected queue length for slow service rate is given by

$$L_q = \sum_{k=1}^{K} k \left[\frac{1}{k!} \prod_{n=1}^{k} \frac{\lambda}{[\mu + (n-1)\upsilon r]} \right] \left(P_0 - \frac{\lambda}{\mu} \right) \tag{17}$$

Expected queue length for fast service rate is given by

$$L_q = \sum_{k=1}^{K} k \left[\frac{1}{k!} \prod_{n=1}^{k} \frac{\lambda}{[\mu_1 + (n-1)\upsilon r]} \right] \left(P_0 - \frac{\lambda}{\mu_1} \right) \tag{18}$$

1.2 EXPECTED SYSTEM SIZE(Ls)

Expected queue length for slow service rate is given by

$$L_s = \sum_{k=1}^{K} k \frac{1}{k!} \prod_{n=1}^{k} \frac{\lambda}{[\mu + (n-1)\upsilon r]} \left(P_0 - \frac{\lambda}{\mu} \right) \tag{19}$$

Expected queue length for fast service rate is given by

$$L_s = \sum_{k=1}^{K} k \frac{1}{k!} \prod_{n=1}^{k} \frac{\lambda}{[\mu_1 + (n-1)\upsilon r]} \left(P_0 - \frac{\lambda}{\mu_1} \right) \tag{20}$$

4.3 EXPECTED WAITING TIME IN THE QUEUE(Wq)

Expected waiting Time in the queue for slow service rate is given by

$$W_q = \left[\frac{1}{\lambda} \sum_{k=1}^{K} k \left(\frac{1}{k!} \prod_{n=1}^{k} \frac{\lambda}{[\mu + (n-1)\upsilon r]} \right) P_0 - \frac{1}{\mu} \right] \tag{21}$$

Expected waiting Time in the queue for fast service rate is given by

$$W_q = \left[\frac{1}{\lambda} \sum_{k=1}^{K} k \left(\frac{1}{k!} \prod_{n=1}^{k} \frac{\lambda}{[\mu_1 + (n-1)\upsilon r]} \right) P_0 - \frac{1}{\mu_1} \right] \tag{22}$$

4.4 EXPECTED WAITING TIME IN THE SYSTEM (Ws)

Expected waiting Time in the system for slow service rate is given by

$$W_s = \left[\frac{1}{\lambda}\sum_{k=1}^{K} k\left(\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{[\mu+(n-1)\upsilon r]}\right)P_0\right] \quad (23)$$

Expected waiting Time in the system for fast service rate is given by

$$W_s = \left[\frac{1}{\lambda}\sum_{k=1}^{K} k\left(\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{[\mu_1+(n-1)\upsilon r]}\right)P_0\right] \quad (24)$$

4.5 EXPECTED NUMBER OF CUSTOMERS SERVED

Expected number of customer served for slow rate given by

$$E(customerserved) = \sum k\mu P_k \quad (25)$$

$$= \sum_{k=1}^{K} k\mu\left[\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{\mu+(n-1)\upsilon r}\right]P_0$$

Expected number of customer served for fast rate given by

$$E(customerserved) = \sum k\mu P_k \quad (26)$$

$$= \sum_{k=1}^{K} k\mu\left[\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{\mu_1+(n-1)\upsilon r}\right]P_0$$

4.6 ABANDONMENT RATE (R_a)

The abandonment rate of customers is given by

$$R_a = \lambda\sum_{k=1}^{K} P_k - E(Customer\,Served)$$

$$R_a = \lambda - \sum_{k=1}^{K} k\mu\left[\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{\mu+(n-1)\upsilon r}\right]P_0 \text{ for the slow service rate} \quad (27)$$

$$R_a = \lambda - \sum_{k=1}^{K} k\mu_1\left[\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{\mu_1+(n-1)\upsilon r}\right]P_0 \text{ for the fast service rate} \quad (28)$$

4.7 EXPECTED NUMBER OF ACTUALLY WAITING CUSTOMERS

The expected number of actually waiting customers is given by

$$E(Customer\,waiting) = \frac{\sum_{k=2}^{K}(k-1)P_k}{\sum_{k=2}^{K}P_k}\lambda\sum_{k=1}^{K}P_k - E(Customer\,Served)$$

$$E(Customer\,waiting) = \frac{\sum_{k=2}^{K}(k-1)\left[\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{\mu+(n-1)\upsilon r}\right]P_0}{\sum_{k=2}^{K}\left[\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{\mu+(n-1)\upsilon r}\right]P_0} \text{ for slow service rate} \quad (29)$$

$$E(Customer\,waiting) = \frac{\sum_{k=2}^{K}(k-1)\left[\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{\mu_1+(n-1)\upsilon r}\right]P_0}{\sum_{k=2}^{K}\left[\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{\mu_1+(n-1)\upsilon r}\right]P_0} \text{ for fast service rate} \quad (30)$$

4.8 PROBABILITY DISTRIBUTION FOR BUSY PERIOD

Probability distribution of busy period is given by

$$P(Busyperiod) = P(k \geq 1) \quad (31)$$

Probability distribution of busy period for slow rate is given by

$$P(Busyperiod) = \sum_{k=1}^{K}\left[\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{\mu+(n-1)\upsilon r}\right]P_0 \quad (32)$$

Probability distribution of busy period for fast rate is given by

$$P(Busyperiod) = \sum_{k=1}^{K}\left[\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{\mu_1+(n-1)\upsilon r}\right]P_0 \quad (33)$$

V.SOME SPECIAL CASES

5.1 THERE IS NO RETENTION OF RENEGED CUSTOMERS

The queueing system is reduced to a system with state dependent arrivals and reneging with slow rate is given by

$$P_n = \left[\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{\mu+(n-1)\upsilon r}\right]P_0\,,\ 1 \le k \le K \quad (34)$$

The queueing system is reduced to a system with state dependent arrivals and reneging with slow rate is given by

$$P_n = \left[\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{\mu_1+(n-1)\upsilon r}\right]P_0\,,\ 1 \le k \le K \quad (35)$$

Using the normalization condition, $\sum_{k=0}^{K} P_k = 1$ we get

$$P_0 = \frac{1}{1+\sum_{k=1}^{K}\left(\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{\mu+(n-1)\upsilon}\right)} \quad \text{for the slow service rate} \quad (36)$$

$$P_0 = \frac{1}{1+\sum_{k=1}^{K}\left(\frac{1}{k!}\prod_{n=1}^{k}\frac{\lambda}{\mu_1+(n-1)\upsilon}\right)} \quad \text{for the fast service rate} \quad (37)$$

5.2 THERE IS NO DISCOURAGEMENT OF CUSTOMERS

In this section, we study two sub cases:

(i) The model reduces to an M/M/1/K queueing system with retention of reneged customers as given below

$$P_k = \prod_{n=1}^{k}\frac{\lambda}{[\mu+(n-1)\upsilon r]}P_0,\ 1 \le k \le K-1 \quad \text{for the slow service rate} \quad (38)$$

$$P_k = \prod_{n=1}^{k}\frac{\lambda}{[\mu_1+(n-1)\upsilon r]}P_0,\ 1 \le k \le K-1 \quad \text{for the fast service rate} \quad (39)$$

For k=K, we get

$$P_K = \prod_{n=1}^{k}\frac{\lambda}{[\mu+(n-1)\upsilon r]}P_0 \quad \text{for the slow service rate} \quad (40)$$

$$P_K = \prod_{n=1}^{k}\frac{\lambda}{[\mu_1+(n-1)\upsilon r]}P_0 \quad \text{for fast service rate} \quad (41)$$

Using the normalization condition, $\sum_{k=0}^{K} P_k = 1$ we get

$$P_0 = \frac{1}{1+\sum_{k=1}^{K}\left(\prod_{n=1}^{k}\frac{\lambda}{\mu+(n-1)\upsilon}\right)} \quad \text{for the slow service rate} \quad (42)$$

for the fast service rate (43)

$$P_0 = \frac{1}{1+\sum_{k=1}^{K}\left(\prod_{n=1}^{k}\frac{\lambda}{\mu_1+(n-1)\upsilon}\right)}$$

(ii) When there is no reneging (i.e. the customers do not get impatient). In this case, the probability of reneging (r) is zero, i.e. reneging rate υ=0. As there is no reneging, so there is no question of customer retention. All the customers who enter into the system leave after getting service. Therefore, in this case

$$P_k = \left(\frac{\lambda}{\mu}\right)^k P_0\,,\ 1\le k\le K \ \text{ for slow service rate} \tag{44}$$

$$P_k = \left(\frac{\lambda}{\mu_1}\right)^k P_0\,,\ 1\le k\le K \ \text{ for fast service rate} \tag{45}$$

Using the normalization condition, $\sum_{k=0}^{K} P_k = 1$ we get

(46)

$$P_0 = \frac{1}{1+\sum_{k=1}^{K}\left(\frac{\lambda}{\mu}\right)^k} \ \text{ for slow service rate}$$

(47)

$$P_0 = \frac{1}{1+\sum_{k=1}^{K}\left(\frac{\lambda}{\mu_1}\right)^k} \ \text{ for fast service rate}$$

VI.CONCLUSION

This paper studies a single server queueing model with detainment of retracted customers in state dependent arrival process. We obtain the steady state solution with some important measures of performance. Special cases of queueing model also discussed. In this paper, the queueing model analysis is limited for finite capacity. The infinite capacity case of the model can also be studied. Model can also be solved in transient state to get time dependent result and cost profit analysis of the model can also be carried out.

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