A New Approach to Number of Spanning Trees of Wheel Graph W_n in Terms of Number of Spanning Trees of Fan Graph F_n

Nihar Ranjan Panda* & Purna Chandra Biswal**

*Research Scholar, P. G. Dept. of Mathematics, Ravenshaw University, Cuttack, Odisha.

**Department of Mathematics, Parala Maharaja Engineering College, Berhampur, Odisha-761003, India.

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ABSTRACT *An exclusive expression τ(Wn) = 3τ(Fn)−2τ(Fn−1)−2 for determining the number of spanning trees of wheel graph Wn is derived using a new approach by defining an onto mapping from the set of all spanning trees of Wn+1 to the set of all spanning trees of Wn.*

Keywords: *Spanning Tree, Wheel, Fan*

1. Introduction

A graph consists of a non-empty vertex set V (G) of n vertices together with a prescribed edge set E(G) of r unordered pairs of distinct vertices of V(G). A graph G is called labeled graph if all the vertices have certain label, and a tree is a connected acyclic graph according to Biggs [1]. All the graphs in this paper are finite, undirected, and simple connected. For a graph G = (V(G), E(G)), a spanning tree in G is a tree which has the same vertex set as G has. The number of spanning trees in a graph G, denoted by τ(G), is an important invariant of the graph. It is also an important measure of reliability of a network. The join of two disjoint graphs H and K, $H_V K$, is obtained from the union of H and K by additionally joining every vertex of H to every vertex of K. The wheel graph $W_{n+1} = C_{n\,V}\, K_1$ where C_n and K_1 are the cycle graph and complete graph respectively. The fan graph $F_{n+1} = P_{n\,V}\, K_1$ where P_n is the path graph. In this paper, the author has derived a closed form formula for the required number of labeled spanning trees of wheel graph W_{n+1}. Again, the solution of the recurrence relation is compared with the solution which is obtain by Sedlacek [2], Myers[3] using the matrix tree theorem and Haghighi & Bibak[4] using mixed recurrence relation containing τ (W_{n+1}) and τ (F_{n+1}).

A famous and classic result on the study of τ (G) is the matrix tree theorem which can be found in Biswal [5]. The Laplacian matrix of a graph G is defined as L(G) = D(G) -A(G) where D(G) and A(G) are the degree matrix and the adjacency matrix of G respectively. The matrix tree theorem states that for any connected graph G, τ (G) is equal to any cofactor of L(G).

2. Recurrence Relation

There is a simple and elegant recursive formula for the number of spanning trees of a graph G= (V (G), E(G)). It involves the operation of contraction of an edge, which now we introduce. An edge $e_{uv} \in E(G)$ is said to be contracted if it is deleted and its vertices are identified. The resulting graph is denoted by $G.e_{uv}$. Also, we denote G - e_{uv} the graph obtained from the graph G by deleting the edge e_{uv}. The corresponding formula is τ (G) = τ (G- e_{uv}) + τ (G.e_{uv}) which can be found in Biswal [5]. This formula is very useful for graph with smaller size of vertex set.

Theorem 1: If τ (W_n) is the number of spanning trees of the wheel graph W_n, then (W_n) = 3 τ (F_n)- 2 τ (F_{n-1}) -2.

Let X be the set of all spanning trees of W_{n+1} and Y be the set of all spanning trees of W_n. Define a mapping Ø : X → Y according to Biswal and Panda [6]. The object and image multiplicity β and α under the mapping Ø : X → Y have to satisfy the relation.

$$\begin{aligned} 24\beta &= 66\alpha - 150 - 7\tau(F_{n-3}) - 6\sum_{i=1}^{n-4}\tau(F_{n-3-i}) - 4\sum_{i=1}^{n-5}\tau(F_{n-3-i}) - 3\sum_{i=2}^{n-4}\binom{i-1}{1}\tau(F_{n-3-i}) \\ &\quad -(\tau(F_{n-3}) - (n-4)) - n \qquad (1) \end{aligned}$$

and

$$24\delta = 66\zeta - 174 - 2 \qquad (2)$$

Again, equation 24 β = 66α 150 has integer solution when α = 3 and β= 2, and equation 24δ = 66 ζ - 174 has integer solution when ζ = 3 and δ = 1. Clearly, the expression which is to be derived is atomic one according to Biswal and Panda[6]. This integer solution and equations (1) and (2) can motivate one to assume that

$$\begin{aligned}\tau(W_{n+1}) &= \alpha\tau(W_n) - \beta\tau(W_{n-1}) + 7\tau(F_{n-3}) + 6\sum_{i=1}^{n-4}\tau(F_{n-3-i}) + 4\sum_{i=1}^{n-5}\tau(F_{n-3-i}) + \\ &\quad 3\sum_{i=2}^{n-4}\binom{i-1}{1}\tau(F_{n-3-i}) + (\tau(F_{n-3}) - (n-4)) + n\end{aligned}$$

and

$$\tau(W_{n+1}) = \zeta\tau(W_n) - \delta\tau(W_{n-1}) + 2$$

because the number of multiple counted over elements in X must be integer multiple of the number of image bundles in U, the set of all spanning trees in W_{n-1}. In other words, the number of multiple counted elements in X must be equal to β $\tau(W_{n-1})$. Hence one can conclude using the properties of $\tau(F_n)$ derived in Biswal and Panda [7] that the recurrence relation for the number of spanning trees of wheel graph W_{n+1} is which is produced by Haghighi & Bibak [4].

$$\begin{aligned}\tau(W_{n+1}) &= 3\tau(W_n) - 2\tau(W_{n-1}) + 8\tau(F_{n-3}) + \sum_{i=1}^{n-5}(7+3i)\tau(F_{n-3-i}) + 3n - 5 \\ &= 3\tau(W_n) - 2\tau(W_{n-1}) + 7\tau(F_{n-2}) - 3\tau(F_{n-3}) \qquad (3)\end{aligned}$$

and

$$\tau(W_{n+1}) = 3\tau(W_n) - \tau(W_{n-1}) + 2 \qquad (4)$$

Using equations (4) and (3), we have

$$\begin{aligned}\tau(W_{n-1}) &= 7\tau(F_{n-2}) - 3\tau(F_{n-3}) - 2 \\ &= 3\tau(F_{n-1}) - 2\tau(F_{n-2}) - 2\end{aligned}$$

using $\tau(F_n) = 3\tau(F_{n-1}) - \tau(F_{n-2})$ given in Biswal and Panda [8].

3. Conclusion

Clearly, expression $\tau(W_{n-1}) = 3\tau(F_{n-1}) - 2\tau(F_{n-2}) - 2$ according to theorem 1 express number of spanning trees of wheel graph W_{n-1} in terms of number of spanning trees of fan graph F_{n-1} and F_{n-2} which is not found by any researcher. Again, one can verify that

$$\begin{aligned}\tau(W_{n-1}) &= 3\tau(F_{n-1}) - 2\tau(F_{n-2}) - 2 \\ &= \left(\frac{3+\sqrt{5}}{2}\right)^{n-2} + \left(\frac{3-\sqrt{5}}{2}\right)^{n-2} - 2\end{aligned}$$

which is produced by Haghighi & Bibak [4].

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