ON ODD PRIME LABELING

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ABSTRACT *This paper introduces a new labeling called an odd prime labeling. Standard graphs like path, complete bipartite graph, wheel and other wheel related graphs including generalized Petersen graph P(n, 2) have been proved to be odd prime in this paper.*

Index Terms: *graph labeling, prime graph, odd prime graph, generalized Petersen graph.*
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1. Introduction

This paper deals with finite, undirected, simple and non-trivial graph G(V, E), with vertex set V or V (G) and edge set E or E(G). The number of elements of V is denoted by |V| and is called the order of graph G. The number of elements of E is denoted by |E| and is called the size of graph G. For graph theoretical notations and terminologies that are not specified here, Clark and Holton [7] is referred. For number theoretical notations, Burton [3] is referred.

This paper is aimed to discuss a new graph labeling technique called odd prime labeling.

We start with a brief summary of definitions and notations required for this paper starting with some standard graphs and graph operations.

Definition 1.1: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then Cartesian product of G_1 and G_2 denoted by $G_1 \times G_2$, is the graph with vertex set $V = V_1 \times V_2$ consisting of vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ such that u and v are adjacent in $G_1 \times G_2$ whenever either $u_1 = v_1$ and u_2 is adjacent to v_2 or u_1 is adjacent to v_1 and $u_2 = v_2$.

Definition 1.2: A ladder graph L_n is the Cartesian product of two paths P_n and P_2. i.e. $P_n \times P_2$.

Definition 1.3: For each graph $n \geq 3$, the wheel graph W_n is defined as the join of the complete graph K_1 and the cycle C_n. i.e. $W_n = C_n + K_1$.

The vertex associated with the complete graph K_1 is called the apex of W_n. The vertices and edges of C_n are respectively called the rim vertices and rim edges of W_n.

Definition 1.4: [6] A gear graph G_n is obtained from the wheel graph W_n by subdividing each rim edge of W_n by a vertex.

Definition 1.5: [6] A helm graph H_n is obtained from the wheel graph W_n by joining a new pendant vertex to each of its rim vertices by an edge. These edges are called pendant edges.

Definition 1.6: [6] A closed helm graph CH_n is obtained from the helm graph H_n by adding edges between the consecutive pendant vertices of H_n.

Definition 1.7: [6] A flower graph Fl_n is obtained from the helm graph H_n by joining each pendant vertex to the apex of H_n .

Definition 1.8: [6] A fan graph f_n is defined to be the join of the complete graph K_1 and path P_n. i.e. $f_n = P_n + K_1$.

Note. We note that f_n can also be obtained by deleting an arbitrary rim edge of W_n.

Definition 1.9: A chord of a cycle C_n is an edge joining two non-adjacent vertices of C_n.

Definition 1.10: Two chords of a cycle C_n ($n \geq 5$) are said to be twin chords if they form a triangle with an edge of the cycle C_n.

Definition 1.11: [6] A shell graph S_n is defined to be a cycle C_n with $n - 3$ concurrent chords. The vertex at which all the chords are concurrent is called the apex. We observe that $S_n = f_{n-1}$.

Definition 1.12: [6] A shell butterfly graph is a one-point union of two shells of any order with two pendent edges at the apex.

Definition 1.13: [6] A shell cactus graph denoted by $S_k^{(n)}$ is a one-point union of n shell graphs S_k at the apex.

Definition 1.14: [6] A friendship graph F_n is a one point union of n cycles C_3.

Definition 1.15: [9] For each graph $n \geq 3$ and $1 \leq k \leq \left[\frac{n}{2}\right]$, the generalized Petersen graph (GPG) denoted by $P(n,\ k)$ is defined to be a graph with vertex set $V = \{u_i,\ v_i \mid i = 1,\ 2, \ldots,\ n\}$ and edge set $E = \{v_i v_{i+1},\ v_i u_i,\ u_i u_{i+k} \mid i = 1,\ 2, \ldots,\ n\}$, where subscripts are taken modulo n. Here v_i's are the outer vertices and u_i's are the inner vertices of $P(n,\ k)$.

The concept of generalized Petersen graphs was given by Watkins [9].

Notation 1.16: [5] We denote $p(a,\ n)$ as the set of all primes x such that $a<x \leq n$.

Notation 1.17: [10] For each natural number n, we denote $[n]$ as the set of first n natural numbers.
i.e. $[n]$ = {1, 2, ...,n}.

Notation 1.18: For each natural number n, we denote O_n as the set of first n odd natural numbers.
i.e. O_n = {1, 3, ..., 2n – 1}.

Definition 1.19: [6] An assignment of integers to vertices and/or edges of a graph, subject to certain conditions is known as graph labeling.

Graph labeling had been introduced in mid 1960. Since then, over 200 graph labeling techniques have been studied in over 2500 papers. For latest survey of graph labeling, Gallian [6] is referred. One such labeling, called prime labeling was originated by Entringer. It was first introduced by Tout et al [2] in their paper.

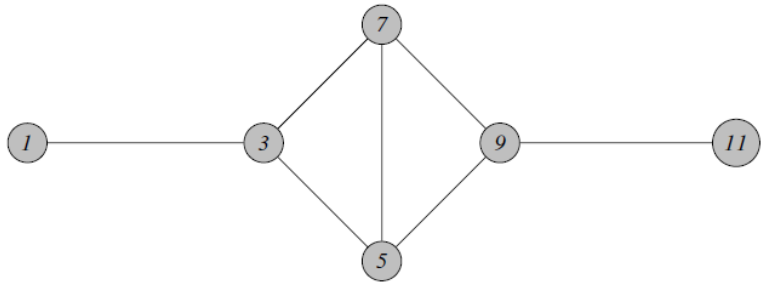
Definition 1.20: [2] Let G(V, E) be a graph. A bijection $f : V \to \{1, 2, \ldots, |V|\}$ is called a **prime labeling** if for each $uv \in E$, $\gcd(f(u), f(v)) = 1$. The graph which admits prime labeling is called a **prime graph**.

Rao [14], Deretsky et al [15], Lee et al [13], Seoud et al [8] in their papers, proved that path P_n, cycle graph C_n, wheel graph W_n if and only if n is even, fan graph f_n, helm graph H_n, flower graph Fl_n admits prime labeling. Vaidya and Prajapati [12] in their paper proved that the graphs obtained by duplication of a vertex by a vertex in P_n and $K_{1,n}$ are prime graphs. They also proved that the graphs obtained by duplication of a vertex by an edge, duplication of an edge by a vertex,duplication of an edge by an edge in P_n, $K_{1,n}$ and C_n are prime graphs. Further they proved that graphs obtained by duplication of every vertex by an edge in P_n, $K_{1,n}$ and C_n are not prime graphs. Prajapati and Gajjar [16] proved that a necessary condition for generalized Petersen graph $P(n, k)$ to be prime is that n is even and k is odd. They also gave some classes of generalized Petersen graphs that admit prime labeling. Fu and Huang [5] in their paper proved that a complete bipartite graph $K_{m,n}$ of order $v = m+n$ with $m \leq n$ is a prime graph iff $m \leq \left| p\left(\frac{v}{2}, v\right)\right|$. They also proved that all trees of order less than 16 and complete binary trees are prime graphs. Berliner et al [1] proved in their paper that ladder graph is a prime graph if n+1 is a prime number. Later Ghorbani and Kamali [4] proved that any ladder graph is a prime graph. Generalizing the concept of prime labeling, notion of k-prime labeling was introduced by Vaidya and Prajapati [11]. They proved that path P_n is a k-prime graph and certain other graphs obtained by identifying a specific vertex in a k_1-prime graph and k_2-prime graph are prime graphs under certain conditions.

Analogous to prime labeling a new labeling called an odd prime labeling is defined as follows:

Definition 1.21: Let G(V, E) be a graph. A bijection $f : V \to O_{|V|}$ is called an ***odd prime labeling*** if for each $uv \in E$, $\gcd(f(u), f(v)) = 1$. The graph which admits odd prime labeling is called an ***odd prime graph***.

Example: A graph with 6 vertices admitting odd prime labeling is shown below:



An immediate trivial result can be noted as follows:

Result 1.22: If G(V, E) is an odd prime graph, and E′ ⊂ E, then G – E′, obtained by deleting all edges of E′ from G is also an odd prime graph.

We note some number theoretical results used in this paper.

Result 1.23: For any odd integer n and positive integer k, $\gcd\left(n, n+2^k\right)=1$.

Result 1.24: For an odd integer n, an odd prime p and positive integers k_1 and k_2, if $n \not\equiv 0 \pmod{p}$ then $\gcd\left(n, n+2^{k_1}\cdot p^{k_2}\right)=1$.

2. Main Results

Theorem 2.1: Path P_n is an odd prime graph.

Proof: Let $v_1, v_2, \ldots, v_n$ be consecutive vertices of a path P_n. So, $\left|V\left(P_n\right)\right|=n$ and $\mathrm{E}\left(P_n\right)=\left\{v_i v_{i+1} \mid i \in [n-1]\right\}$. Define $f: \mathrm{V}\left(P_n\right) \to O_n$ as $f\left(v_i\right)=2i-1,\ i \in [n]$. Clearly, for an arbitrary edge $v_i v_{i+1} \in E$, $\gcd\left(f\left(v_i\right), f\left(v_{i+1}\right)\right)$ = $\gcd(2i-1,\ 2i+1)=1$. Thus f admits odd prime labeling on P_n and hence P_n is an odd prime graph.

Theorem 2.2: Ladder graph L_n is an odd prime graph.

Proof: Let two copies of a path P_n have vertices $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$. Let L_n be the ladder graph obtained by joining u_i and v_i by an edge for each $i \in [n]$. Hence, $\left|\mathrm{V}\left(L_n\right)\right|=2n$ and $\mathrm{E}\left(L_n\right)=\left\{u_i u_{i+1}, v_i v_{i+1}, u_i v_i \,\middle|\, i \in [n-1]\right\} \cup \left\{u_n v_n\right\}$. Define $f: \mathrm{V} \to \mathrm{O}_{2n}$ as $f(x)=\begin{cases} 4i-3, & \text{if } x=u_i,\ i \in [n]; \\ 4i-1, & \text{if } x=v_i,\ i \in [n]. \end{cases}$

With this labeling for each e of graph L_n,

1. If $e=u_i u_{i+1},\ i \in [n-1]$ then $\gcd\left(f\left(u_i\right), f\left(u_{i+1}\right)\right)=\gcd(4i-3,\ 4i+1)=1$.
2. If $e=v_i v_{i+1},\ i \in [n-1]$ then $\gcd\left(f\left(v_i\right), f\left(v_{i+1}\right)\right)=\gcd(4i-1,\ 4i+3)=1$.
3. If $e=u_i v_i,\ i \in [n]$ then $\gcd\left(f\left(u_i\right), f\left(v_i\right)\right)=\gcd(4i-3,\ 4i-1)=1$.

Thus f admits odd prime labeling on L_n and hence L_n is an odd prime graph.

Theorem 2.3: Complete bipartite graph $K_{m,n}$ is an odd prime graph iff $k \le \left|p\left(\frac{2m+2n-1}{3}, 2m+2n-1\right)\right|+1$, where $k \in \{m,\ n\}$.

Proof: Let V = X ∪ Y be the bipartition of the complete bipartite graph $K_{m,n}$, where $\mathrm{X}=\{u_1, u_2, \ldots, u_m\}$ and $\mathrm{Y}=\{v_1, v_2, \ldots, v_n\}$. So, $|\mathrm{V}|=m+n$. Let $p\left(\frac{2m+2n-1}{3}, 2m+2n-1\right) \cup \{1\}$ be of order t given by $\{p_1, p_2, \ldots, p_t\}$ where $p_1=1$.

Without loss of generality, assume $m \le \left| p\left(\frac{2m+2n-1}{3}, 2m+2n-1\right)\right| + 1 = t$.

Define a function $f : \mathrm{V} \to \mathrm{O}_{m+n}$ as $f(x) = \begin{cases} p_i, \text{ if } x = u_i,\ i \in [m]; \\ q_j, \text{ if } x = v_j,\ j \in [n], \end{cases}$ where $\{q_j \mid j \in [n]\} = \mathrm{O}_{m+n} - \{f(u_i) \mid i \in [m]\}$.

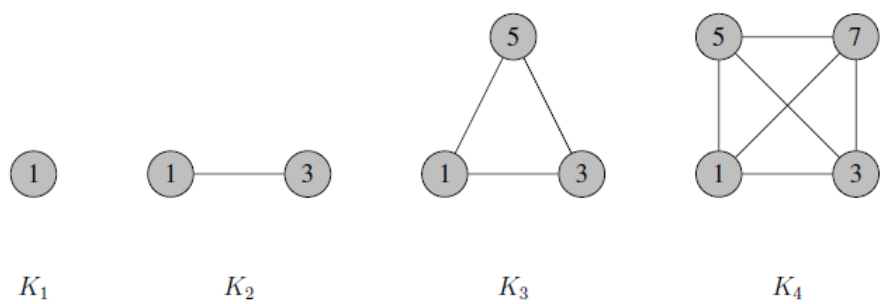
As $m \le t$, f is a well defined function and it can easily be verified that f is a bijection.

Now for any $x \in p\left(\frac{2m+2n-1}{3}, 2m+2n-1\right) \cup \{1\}$, $\gcd(x,i) = 1$ for all $i \in \mathrm{O}_{m+n} - \{x\}$. Thus for any $u_i v_j \in \mathrm{E}(K_{m,n})$, $i \in [m]; j \in [n]$, we have $\gcd(f(u_i), f(v_j)) = 1$. Thus f admits an odd prime labeling on $K_{m,n}$ and hence $K_{m,n}$ is an odd prime graph. Conversely, if $K_{m,n}$ is an odd prime graph, then there exists a bijection $f : \mathrm{V} \to \mathrm{O}_{m+n}$ such that for any $u_i v_j \in \mathrm{E}(K_{m,n})$, $i \in [m]; j \in [n]$, $\gcd(f(u_i), f(v_j)) = 1$.

Since $p\left(\frac{2m+2n-1}{3}, 2m+2n-1\right) \cup \{1\}$ are the only numbers which are relatively prime to all other numbers in the set O_{m+n} we must have, $k \le \left| p\left(\frac{2m+2n-1}{3}, 2m+2n-1\right)\right| + 1$, where $k \in \{m, n\}$.

Theorem 2.4: Complete graph K_n is an odd prime graph iff $n \le 4$.

Proof: Complete graphs K_1, K_2, K_3 and K_4 has odd prime labeling as seen from the figure below.



Now, for any bijective function $f : \mathrm{V}(K_n) \to \mathrm{O}_n$ and $n \ge 5$, we have 3, 9 $\in O_n$. K_n being complete graph, we always have an edge whose end vertices have labels 3 and 9. Since $\gcd(3, 9) \ne 1$, f does not admit odd prime labeling on K_n and hence K_n is not an odd prime graph if $n \ge 5$.

Theorem 2.5: Wheel graph W_n is an odd prime graph.

Proof: Let $v_1, v_2, \ldots, v_n$ be consecutive rim vertices and v be the apex vertex of a wheel graph W_n. So, $|\mathrm{V}(W_n)| = n+1$ and $\mathrm{E}(W_n) = \{vv_i \mid i \in [n]\} \cup \{v_i v_{i+1} \mid i \in [n-1]\} \cup \{v_1 v_n\}$. Now, we have the following cases:

1. If $2n+1 \not\equiv 0 \pmod 3$, define $f : \mathrm{V}(W_n) \to \mathrm{O}_{n+1}$ as: $f(x) = \begin{cases} 1 \quad, \text{ if } x = v; \\ 2i+1, \text{ if } x = v_i,\ i \in [n]. \end{cases}$

   Then for each edge e of W_n:

   (i) If $e = vv_i$, $i \in [n]$ then $\gcd(f(v), f(v_i)) = \gcd(1, 2i+1) = 1$.

   (ii) If $e = v_i v_{i+1}$, $i \in [n-1]$ then $\gcd(f(v_i), f(v_{i+1})) = \gcd(2i+1, 2i+3) = 1$.

   (iii) If $e = v_1 v_n$, then $\gcd(f(v_1), f(v_n)) = \gcd(3, 2n+1) = 1$.

2. If $2n+1 \equiv 0 \pmod 3$, then $2n-1 \not\equiv 0 \pmod 3$. Thus, define $f : \mathrm{V}(W_n) \to \mathrm{O}_{n+1}$ as:

$$f(x) = \begin{cases} 1 \ , \text{ if } x = v; \\ 2i+1, \text{ if } x = v_i, \ i \in [n-2]; \\ 2n+1, \text{ if } x = v_{n-1}; \\ 2n-1, \text{ if } x = v_n. \end{cases}$$

Then for each edge e of W_n:

(i) If $e = vv_i$, $i \in [n]$ then $\gcd(f(v), f(v_i)) = \gcd(1, f(v_i)) = 1$.

(ii) If $e = v_i v_{i+1}$, $i \in [n-3]$ then $\gcd(f(v_i), f(v_{i+1})) = \gcd(2i+1, 2i+3) = 1$.

(iii) If $e = v_{n-2}v_{n-1}$, then $\gcd(f(v_{n-2}), f(v_{n-1})) = \gcd(2n-3, 2n+1) = 1$.

(iv) If $e = v_{n-1}v_n$, then $\gcd(f(v_{n-1}), f(v_n)) = \gcd(2n+1, 2n-1) = 1$.

(v) If $e = v_1 v_n$, then $\gcd(f(v_1), f(v_n)) = \gcd(3, 2n-1) = 1$.

Thus from each of the cases, f admits odd prime labeling on W_n. Hence, W_n is an odd prime graph.

Corollary 2.6: Fan graph f_n, friendship graph F_n shell graph $S_n = f_{n-1}$, shell butterfly graph, shell cactus graph $S_k^{(n)}$ are all odd prime graphs.

In particular,

Corollary 2.7: Cycle graph C_n with one chord, cycle graph C_n with twin chord and cycle graph C_n itself are all odd prime graphs.

Theorem 2.8: Gear graph $G_n$is an odd prime graph.

Proof: Consider the wheel graph W_n with apex vertex v and consecutive rim vertices $v_1, v_2, \ldots, v_n$. Let G_n be the gear graph obtained by subdividing each rim edge of W_n by vertices $u_1, u_2, \ldots, u_n$, where each u_i is added between v_i and v_{i+1} for each $i \in [n-1]$ and u_n is added between v_1 and v_n. So, $|\mathrm{V}(G_n)| = 2n+1$ and $\mathrm{E}(G_n) = \{vv_i, v_i u_i \mid i \in [n]\} \cup \{u_i v_{i+1} \mid i \in [n-1]\} \cup \{u_n v_1\}$. Now, we have the following cases:

1. If $4n+1 \not\equiv 0 \pmod 3$, define $f : \mathrm{V}(G_n) \to \mathrm{O}_{2n+1}$ as: $f(x) = \begin{cases} 1 \ , \text{ if } x = v; \\ 4i-1, \text{ if } x = v_i, \ i \in [n]; \\ 4i+1, \text{ if } x = u_i, \ i \in [n]. \end{cases}$

Then for each edge e of G_n:

(i) If $e = vv_i$, $i \in [n]$ then $\gcd(f(v), f(v_i)) = \gcd(1, 4i-1) = 1$.

(ii) If $e = v_i u_i$, $i \in [n]$ then $\gcd(f(v_i), f(u_i)) = \gcd(4i-1, 4i+1) = 1$.

(iii) If $e = u_i v_{i+1}$, $i \in [n-1]$ then $\gcd(f(u_i), f(v_{i+1})) = \gcd(4i+1, 4i+3) = 1$.

(iv) If $e = u_n v_1$, then $\gcd(f(u_n), f(v_1)) = \gcd(4n+1, 3) = 1$.

2. If $4n+1 \equiv 0 \pmod 3$, then $4n-1 \not\equiv 0 \pmod 3$. Thus, define $f : \mathrm{V}(G_n) \to \mathrm{O}_{2n+1}$ as:

$$f(x) = \begin{cases} 1 \ , \text{ if } x = v; \\ 4i-1, \text{ if } x = v_i, \ i \in [n-1]; \\ 4n+1, \text{ if } x = v_n; \\ 4i+1, \text{ if } x = u_i, \ i \in [n-1]; \\ 4n-1, \text{ if } x = u_n. \end{cases}$$

Then for each edge e of G_n:

(i) If $e = vv_i$, $i \in [n]$ then $\gcd(f(v), f(v_i)) = \gcd(1, f(v_i)) = 1$.

(ii) If $e = v_i u_i$, $i \in [n]$ then $\gcd(f(v_i), f(u_i)) = \gcd(4i-1, 4i+1) = 1$.

(iii) If $e = u_i v_{i+1}$, $i \in [n-2]$ then $\gcd(f(u_i), f(v_{i+1})) = \gcd(4i+1, 4i+3) = 1$.

(iv) If $e = u_{n-1} v_n$, then $\gcd(f(u_{n-1}), f(v_n)) = \gcd(4n-3, 4n+1) = 1$.

(v) If $e = u_n v_1$, then $\gcd(f(u_n), f(v_1)) = \gcd(4n-1, 3) = 1$.

Thus from each of the cases, *f* admits odd prime labeling on G_n. Hence, G_n is an odd prime graph.

Theorem 2.9: Flower graph Fl_n is an odd prime graph.

Proof: Consider the flower graph Fl_n obtained from the Helm graph H_n with apex vertex *v*, consecutive rim vertices $v_1, v_2, \ldots, v_n$ and corresponding pendant vertices $u_1, u_2, \ldots, u_n$. So $|V(Fl_n)| = 2n+1$ and $E(Fl_n) = \{vv_i, v_i u_i, vu_i \mid i \in [n]\} \cup \{v_i v_{i+1} \mid i \in [n-1]\} \cup \{v_n v_1\}$. Now, we have the following cases:

1. If $4n-1 \not\equiv 0 \pmod 3$, define $f : V(Fl_n) \to O_{2n+1}$ as: $f(x) = \begin{cases} 1 & \text{, if } x = v; \\ 4i-1, & \text{if } x = v_i,\ i \in [n]; \\ 4i+1, & \text{if } x = u_i,\ i \in [n]. \end{cases}$

Then for each edge *e* of Fl_n:

(i) If $e = vv_i, vu_i$, $i \in [n]$, then since $f(v) = 1$, $\gcd(f(v), f(v_i)) = \gcd(f(v), f(u_i)) = 1$.

(ii) If $e = v_i u_i$, $i \in [n]$ then $\gcd(f(v_i), f(u_i)) = \gcd(4i-1, 4i+1) = 1$.

(iii) If $e = v_i v_{i+1}$, $i \in [n-1]$ then $\gcd(f(v_i), f(v_{i+1})) = \gcd(4i-1, 4i+3) = 1$.

(iv) If $e = v_n v_1$, then $\gcd(f(v_n), f(v_1)) = \gcd(4n-1, 3) = 1$.

2. If $4n-1 \equiv 0 \pmod 3$, then $4n+1 \not\equiv 0 \pmod 3$. Thus, define $f : V(Fl_n) \to O_{2n+1}$ as:

$$f(x) = \begin{cases} 1 & \text{, if } x = v; \\ 4i-1, & \text{if } x = v_i,\ i \in [n-1]; \\ 4n+1, & \text{if } x = v_n; \\ 4i+1, & \text{if } x = u_i,\ i \in [n-1]; \\ 4n-1, & \text{if } x = u_n. \end{cases}$$

Then for each edge *e* of Fl_n:

(i) If $e = vv_i, vu_i$, $i \in [n]$, then since $f(v) = 1$, $\gcd(f(v), f(v_i)) = \gcd(f(v), f(u_i)) = 1$.

(ii) If $e = v_i u_i$, $i \in [n]$ then $\gcd(f(v_i), f(u_i)) = \gcd(4i-1, 4i+1) = 1$.

(iii) If $e = v_i v_{i+1}$, $i \in [n-2]$ then $\gcd(f(v_i), f(v_{i+1})) = \gcd(4i-1, 4i+3) = 1$.

(iv) If $e = v_{n-1} v_n$, then $\gcd(f(v_{n-1}), f(v_n)) = \gcd(4n-5, 4n+1) = 1$

(v) If $e = v_n v_1$, then $\gcd(f(v_n), f(v_1)) = \gcd(4n+1, 3) = 1$.

Thus from each of the cases, *f* admits odd prime labeling on Fl_n. Hence, Fl_n is an odd prime graph.

Theorem 2.10: Closed Helm graph CH_n is an odd prime graph.

Proof: Let CH_n be a closed helm graph obtained from the helm graph H_n with apex vertex *v*, consecutive rim vertices $v_1, v_2, \ldots, v_n$ and corresponding pendant vertices $u_1, u_2, \ldots, u_n$. So $|V(CH_n)| = 2n+1$ and $E(CH_n) = \{vv_i, v_i u_i \mid i \in [n]\} \cup \{v_i v_{i+1}, u_i u_{i+1} \mid i \in [n-1]\} \cup \{v_n v_1, u_n u_1\}$. Now, we have the following cases:

1. If $4n-1 \not\equiv 0 \pmod 3$ and $4n+1 \not\equiv 0 \pmod 5$, define $f : V(CH_n) \to O_{2n+1}$ as:

$$f(x) = \begin{cases} 1 & \text{, if } x = v; \\ 4i-1, & \text{if } x = v_i,\ i \in [n]; \\ 4i+1, & \text{if } x = u_i,\ i \in [n]. \end{cases}$$

Then for each edge e of CH_n:

(i) If $e = vv_i$, $i \in [n]$, then $\gcd(f(v), f(v_i)) = \gcd(1, 4i-1) = 1$.

(ii) If $e = u_i v_i$, $i \in [n]$ then $\gcd(f(u_i), f(v_i)) = \gcd(4i+1, 4i-1) = 1$.

(iii) If $e = v_i v_{i+1}$, $i \in [n-1]$ then $\gcd(f(v_i), f(v_{i+1})) = \gcd(4i-1, 4i+3) = 1$.

(iv) If $e = v_n v_1$, then $\gcd(f(v_n), f(v_1)) = \gcd(4n-1, 3) = 1$.

(v) If $e = u_i u_{i+1}$, $i \in [n-1]$ then $\gcd(f(u_i), f(u_{i+1})) = \gcd(4i+1, 4i+5) = 1$.

(vi) If $e = u_n u_1$, then $\gcd(f(u_n), f(u_1)) = \gcd(4n+1, 5) = 1$.

2. If $4n-1 \equiv 0 \pmod 3$ and $4n+1 \equiv 0 \pmod 5$, then $4n+1 \not\equiv 0 \pmod 3$ and $4n-1 \not\equiv 0 \pmod 5$.

Define $f : V(CH_n) \to O_{2n+1}$ as: $f(x) = \begin{cases} 1 \ , \text{ if } x = v; \\ 4i-1, \text{ if } x = v_i, \ i \in [n-1]; \\ 4n+1, \text{ if } x = v_n; \\ 4i+1, \text{ if } x = u_i, \ i \in [n-1]; \\ 4n-1, \text{ if } x = u_n. \end{cases}$

Then for each edge e of CH_n:

(i) If $e = vv_i$, $i \in [n]$, then $\gcd(f(v), f(v_i)) = \gcd(1, f(v_i)) = 1$.

(ii) If $e = u_i v_i$, $i \in [n]$ then $\gcd(f(u_i), f(v_i)) = \gcd(4i+1, 4i-1) = 1$.

(iii) If $e = v_i v_{i+1}$, $i \in [n-2]$ then $\gcd(f(v_i), f(v_{i+1})) = \gcd(4i-1, 4i+3) = 1$.

(iv) If $e = v_{n-1} v_n$, then $\gcd(f(v_{n-1}), f(v_n)) = \gcd(4n-5, 4n+1) = 1$.

(v) If $e = v_n v_1$, then $\gcd(f(v_n), f(v_1)) = \gcd(4n+1, 3) = 1$.

(vi) If $e = u_i u_{i+1}$, $i \in [n-2]$ then $\gcd(f(u_i), f(u_{i+1})) = \gcd(4i+1, 4i+5) = 1$.

(vii) If $e = u_{n-1} u_n$, then $\gcd(f(u_{n-1}), f(u_n)) = \gcd(4n-3, 4n-1) = 1$.

(viii) If $e = u_n u_1$, then $\gcd(f(u_n), f(u_1)) = \gcd(4n-1, 5) = 1$.

3. If $4n-1 \equiv 0 \pmod 3$ and $4n+1 \not\equiv 0 \pmod 5$, then $4n-5 \not\equiv 0 \pmod 3$.

Now, we further divide this into two subcases:

(i) $4n-1 \not\equiv 0 \pmod 5$. In this case with the labeling defined as in case 2, f admits odd prime labeling on CH_n.

(ii) $4n-1 \equiv 0 \pmod 5$. In this case define $f : V(CH_n) \to O_{2n+1}$ as:

$$f(x) = \begin{cases} 1 \ , \text{ if } x = v; \\ 4i-1, \text{ if } x = v_i, \ i \in [n-2]; \\ 4n-1, \text{ if } x = v_{n-1}; \\ 4n-5, \text{ if } x = v_n; \\ 4i+1, \text{ if } x = u_i, \ i \in [n]. \end{cases}$$

Then for each edge e of CH_n:

(a) If $e = vv_i$, $i \in [n]$, then $\gcd(f(v), f(v_i)) = \gcd(1, f(v_i)) = 1$.

(b) If $e = u_i v_i$, $i \in [n-2]$ then $\gcd(f(u_i), f(v_i)) = \gcd(4i+1, 4i-1) = 1$.

(c) If $e = u_{n-1} v_{n-1}$, then $\gcd(f(u_{n-1}), f(v_{n-1})) = \gcd(4n-3, 4n-1) = 1$.

(d) If $e = u_n v_n$, then $\gcd(f(u_n), f(v_n)) = \gcd(4n+1, 4n-5) = 1$.

(e) If $e = v_i v_{i+1}$, $i \in [n-3]$ then $\gcd(f(v_i), f(v_{i+1})) = \gcd(4i-1, 4i+3) = 1$.

(f) If $e = v_{n-2} v_{n-1}$, then $\gcd(f(v_{n-2}), f(v_{n-1})) = \gcd(4n-9, 4n-1) = 1$.

(g) If $e = v_{n-1}v_n$, then $\gcd\left(f(v_{n-1}),\ f(v_n)\right) = \gcd(4n-1,\ 4n-5) = 1$.

(h) If $e = v_n v_1$, then $\gcd\left(f(v_n),\ f(v_1)\right) = \gcd(4n-5,\ 3) = 1$.

(i) If $e = u_i u_{i+1}$, $i \in [n-1]$ then $\gcd\left(f(u_i),\ f(u_{i+1})\right) = \gcd(4i+1,\ 4i+5) = 1$.

(j) If $e = u_n u_1$, then $\gcd\left(f(u_n),\ f(u_1)\right) = \gcd(4n+1,\ 5) = 1$.

Thus in this case, *f* admits odd prime labeling on CH_n.

4. If $4n-1 \not\equiv 0 \pmod 3$ and $4n+1 \equiv 0 \pmod 5$, then $4n-3 \not\equiv 0 \pmod 5$. Now, we further divide this into two subcases:

   (i) $4n+1 \not\equiv 0 \pmod 3$. In this case with the labeling defined as in case 2, *f* admits odd prime labeling on CH_n.

   (ii) $4n+1 \equiv 0 \pmod 3$. Thus $4n-3 \not\equiv 0 \pmod 3$. In this case define $f : \mathrm{V}(CH_n) \to \mathrm{O}_{2n+1}$ as:

$$f(x) = \begin{cases} 1 \ , \text{ if } x = v; \\ 4i-1, \text{ if } x = v_i,\ i \in [n-3]; \\ 4n-5, \text{ if } x = v_{n-2}; \\ 4n-1, \text{ if } x = v_{n-1}; \\ 4n-9, \text{ if } x = v_n; \\ 4i+1, \text{ if } x = u_i,\ i \in [n-2]; \\ 4n+1, \text{ if } x = u_{n-1}; \\ 4n-3, \text{ if } x = u_n. \end{cases}$$

Then for each edge *e* of CH_n:

(a) If $e = vv_i$, $i \in [n]$, then $\gcd\left(f(v),\ f(v_i)\right) = \gcd\left(1,\ f(v_i)\right) = 1$.

(b) If $e = u_i v_i$, $i \in [n-3]$ then $\gcd\left(f(u_i),\ f(v_i)\right) = \gcd(4i+1,\ 4i-1) = 1$.

(c) If $e = u_{n-2}v_{n-2}$, then $\gcd\left(f(u_{n-2}),\ f(v_{n-2})\right) = \gcd(4n-7,\ 4n-5) = 1$.

(d) If $e = u_{n-1}v_{n-1}$, then $\gcd\left(f(u_{n-1}),\ f(v_{n-1})\right) = \gcd(4n+1,\ 4n-1) = 1$.

(e) If $e = u_n v_n$, then $\gcd\left(f(u_n),\ f(v_n)\right) = \gcd(4n-3,\ 4n-9) = 1$.

(f) If $e = v_i v_{i+1}$, $i \in [n-4]$ then $\gcd\left(f(v_i),\ f(v_{i+1})\right) = \gcd(4i-1,\ 4i+3) = 1$.

(g) If $e = v_{n-3}v_{n-2}$, then $\gcd\left(f(v_{n-3}),\ f(v_{n-2})\right) = \gcd(4n-13,\ 4n-5) = 1$

(h) If $e = v_{n-2}v_{n-1}$, then $\gcd\left(f(v_{n-2}),\ f(v_{n-1})\right) = \gcd(4n-5,\ 4n-1) = 1$.

(i) If $e = v_{n-1}v_n$, then $\gcd\left(f(v_{n-1}),\ f(v_n)\right) = \gcd(4n-1,\ 4n-9) = 1$.

(j) If $e = v_n v_1$, then $\gcd\left(f(v_n),\ f(v_1)\right) = \gcd(4n-9,\ 3) = 1$.

(k) If $e = u_i u_{i+1}$, $i \in [n-3]$ then $\gcd\left(f(u_i),\ f(u_{i+1})\right) = \gcd(4i+1,\ 4i+5) = 1$.

(l) If $e = u_{n-2}u_{n-1}$, then $\gcd\left(f(u_{n-2}),\ f(u_{n-1})\right) = \gcd(4n-7,\ 4n+1) = 1$.

(m) If $e = u_{n-1}u_n$, then $\gcd\left(f(u_{n-1}),\ f(u_n)\right) = \gcd(4n+1,\ 4n-3) = 1$.

(n) If $e = u_n u_1$, then $\gcd\left(f(u_n),\ f(u_1)\right) = \gcd(4n-3,\ 5) = 1$.

From each of the cases, we see that *f* admits odd prime labeling on CH_n and hence CH_n is an odd prime graph.

Corollary 2.11: Helm graph H_n is an odd prime graph.

Theorem 2.12: A generalized Petersen graph $P(n, 2)$, is an odd prime graph with $n \geq 3$.

Proof: Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be respectively the inner and outer consecutive vertices of $P(n, 2)$. So $\left|\mathrm{V}\left(P(n,2)\right)\right| = 2n$ and $\mathrm{E}\left(P(n,2)\right) = \left\{u_i v_i, v_i v_{i+1}, u_i u_{i+2} \,\middle|\, i \in [n]\right\}$, where all subscripts are taken modulo n. We consider the following cases:

1. If $4n-1 \not\equiv 0 \pmod 3$ and $4n-3 \not\equiv 0 \pmod 5$, define $f : \mathrm{V}\left(P(n,2)\right) \to \mathrm{O}_{2n}$ as:

$$f(x) = \begin{cases} 4i-3, \text{ if } x = u_i,\ i \in [n]; \\ 4i-1, \text{ if } x = v_i,\ i \in [n]. \end{cases}$$

Then for each edge e of $P(n, 2)$:

(i) If $e = u_i v_i$, $i \in [n]$, then $\gcd\left(f(u_i), f(v_i)\right) = \gcd(4i-3,\ 4i-1) = 1$.

(ii) If $e = v_i v_{i+1}$, $i \in [n-1]$ then $\gcd\left(f(v_i), f(v_{i+1})\right) = \gcd(4i-1,\ 4i+3) = 1$.

(iii) If $e = v_n v_1$, then $\gcd\left(f(v_n), f(v_1)\right) = \gcd(4n-1,\ 3) = 1$.

(iv) If $e = u_i u_{i+2}$, $i \in [n-2]$ then $\gcd\left(f(u_i), f(u_{i+2})\right) = \gcd(4i-3,\ 4i+5) = 1$.

(v) If $e = u_{n-1} u_1$, then $\gcd\left(f(u_{n-1}), f(u_1)\right) = \gcd(4n-7,\ 1) = 1$.

(vi) If $e = u_n u_2$, then $\gcd\left(f(u_n), f(u_2)\right) = \gcd(4n-3,\ 5) = 1$.

2. If $4n-1 \equiv 0 \pmod 3$ and $4n-3 \equiv 0 \pmod 5$, then $4n-3 \not\equiv 0 \pmod 3$ and $4n-1 \not\equiv 0 \pmod 5$.

Now define $f : \mathrm{V}\left(P(n,2)\right) \to \mathrm{O}_{2n}$ as: $f(x) = \begin{cases} 4i-3, \text{ if } x = u_i,\ i \in [n-1]; \\ 4n-1, \text{ if } x = u_n; \\ 4i-1, \text{ if } x = v_i,\ i \in [n-1]; \\ 4n-3, \text{ if } x = v_n. \end{cases}$

Then for each edge e of $P(n, 2)$:

(i) If $e = u_i v_i$, $i \in [n]$, then $\gcd\left(f(u_i), f(v_i)\right) = \gcd(4i-3,\ 4i-1) = 1$.

(ii) If $e = v_i v_{i+1}$, $i \in [n-2]$ then $\gcd\left(f(v_i), f(v_{i+1})\right) = \gcd(4i-1,\ 4i+3) = 1$.

(iii) If $e = v_{n-1} v_n$, then $\gcd\left(f(v_{n-1}), f(v_n)\right) = \gcd(4n-5,\ 4n-3) = 1$

(iv) If $e = v_n v_1$, then $\gcd\left(f(v_n), f(v_1)\right) = \gcd(4n-3,\ 3) = 1$.

(v) If $e = u_i u_{i+2}$, $i \in [n-3]$ then $\gcd\left(f(u_i), f(u_{i+2})\right) = \gcd(4i-3,\ 4i+5) = 1$.

(vi) If $e = u_{n-2} u_n$, then $\gcd\left(f(u_{n-2}), f(u_n)\right) = \gcd(4n-11,\ 4n-1) = 1$.

(vii) If $e = u_{n-1} u_1$, then $\gcd\left(f(u_{n-1}), f(u_1)\right) = \gcd(4n-7,\ 1) = 1$

(viii) If $e = u_n u_2$, then $\gcd\left(f(u_n), f(u_2)\right) = \gcd(4n-1,\ 5) = 1$.

3. If $4n-1 \equiv 0 \pmod 3$ and $4n-3 \not\equiv 0 \pmod 5$, then $4n-3 \not\equiv 0 \pmod 3$. We further divide this into two subcases:

(i) $4n-1 \not\equiv 0 \pmod 5$. In this case with the labeling defined as in case 2, f admits odd prime labeling on $P(n, 2)$.

(ii) $4n-1 \equiv 0 \pmod 5$. Here, we define $f : \mathrm{V}\left(P(n,2)\right) \to \mathrm{O}_{2n}$ as

$$f(x) = \begin{cases} 4i-3, \text{ if } x = u_i,\ i \in [n-2]; \\ 4n-3, \text{ if } x = u_{n-1}; \\ 4n-7, \text{ if } x = u_n; \\ 4i-1, \text{ if } x = v_i,\ i \in [n-2]; \\ 4n-1, \text{ if } x = v_{n-1}; \\ 4n-5, \text{ if } x = v_n. \end{cases}$$

Then for each edge e of $P(n, 2)$:

(a) If $e = u_i v_i$, $i \in [n]$, then $\gcd(f(u_i), f(v_i)) = \gcd(4i-3, 4i-1) = 1$.

(b) If $e = v_i v_{i+1}$, $i \in [n-3]$ then $\gcd(f(v_i), f(v_{i+1})) = \gcd(4i-1, 4i+3) = 1$.

(c) If $e = v_{n-2} v_{n-1}$, then $\gcd(f(v_{n-2}), f(v_{n-1})) = \gcd(4n-9, 4n-1) = 1$.

(d) If $e = v_{n-1} v_n$, then $\gcd(f(v_{n-1}), f(v_n)) = \gcd(4n-1, 4n-5) = 1$.

(e) If $e = v_n v_1$, then $\gcd(f(v_n), f(v_1)) = \gcd(4n-5, 3) = 1$.

(f) If $e = u_i u_{i+2}$, $i \in [n-4]$ then $\gcd(f(u_i), f(u_{i+2})) = \gcd(4i-3, 4i+5) = 1$.

(g) If $e = u_{n-3} u_{n-1}$, then $\gcd(f(u_{n-3}), f(u_{n-1})) = \gcd(4n-15, 4n-3) = 1$.

(h) If $e = u_{n-2} u_n$, then $\gcd(f(u_{n-2}), f(u_n)) = \gcd(4n-11, 4n-7) = 1$.

(i) If $e = u_{n-1} u_1$, then $\gcd(f(u_{n-1}), f(u_1)) = \gcd(4n-3, 1) = 1$.

(j) If $e = u_n u_2$, then $\gcd(f(u_n), f(u_2)) = \gcd(4n-7, 5) = 1$.

4. If $4n-1 \not\equiv 0 \pmod 3$ and $4n-3 \equiv 0 \pmod 5$, then $4n-1 \not\equiv 0 \pmod 5$. We further divide this into two subcases:

(i) $4n-3 \not\equiv 0 \pmod 3$. In this case with the labeling defined as in case 2, *f* admits odd prime labeling on *P*(*n*, 2).

(ii) $4n-3 \equiv 0 \pmod 3$. Here, we define $f : V(P(n,2)) \to O_{2n}$ as:

$$f(x) = \begin{cases} 4i-3, \text{ if } x = u_i,\ i \in [n-3] \cup \{n-1\}; \\ 4n-3, \text{ if } x = u_{n-2}; \\ 4n-11, \text{ if } x = u_n; \\ 4i-1, \text{ if } x = v_i,\ i \in [n-3] \cup \{n\}; \\ 4n-5, \text{ if } x = v_{n-2}; \\ 4n-5, \text{ if } x = v_{n-1}. \end{cases}$$

Then for each edge *e* of *P*(*n*, 2):

(a) If $e = u_i v_i$, $i \in [n-3]$, then $\gcd(f(u_i), f(v_i)) = \gcd(4i-3, 4i-1) = 1$.

(b) If $e = u_{n-2} v_{n-2}$, then $\gcd(f(u_{n-2}), f(v_{n-2})) = \gcd(4n-3, 4n-5) = 1$.

(c) If $e = u_{n-1} v_{n-1}$, then $\gcd(f(u_{n-1}), f(v_{n-1})) = \gcd(4n-7, 4n-9) = 1$.

(d) If $e = u_n v_n$, then $\gcd(f(u_n), f(v_n)) = \gcd(4n-11, 4n-1) = 1$.

(e) If $e = v_i v_{i+1}$, $i \in [n-4]$ then $\gcd(f(v_i), f(v_{i+1})) = \gcd(4i-1, 4i+3) = 1$.

(f) If $e = v_{n-3} v_{n-2}$, then $\gcd(f(v_{n-3}), f(v_{n-2})) = \gcd(4n-13, 4n-5) = 1$.

(g) If $e = v_{n-2} v_{n-1}$, then $\gcd(f(v_{n-2}), f(v_{n-1})) = \gcd(4n-5, 4n-9) = 1$.

(h) If $e = v_{n-1} v_n$, then $\gcd(f(v_{n-1}), f(v_n)) = \gcd(4n-9, 4n-1) = 1$.

(i) If $e = v_n v_1$, then $\gcd(f(v_n), f(v_1)) = \gcd(4n-1, 3) = 1$.

(j) If $e = u_i u_{i+2}$, $i \in [n-5] \cup \{n-3\}$ then $\gcd(f(u_i), f(u_{i+2})) = \gcd(4i-3, 4i+5) = 1$.

(k) If $e = u_{n-4} u_{n-2}$, then $\gcd(f(u_{n-4}), f(u_{n-2})) = \gcd(4n-19, 4n-3) = 1$.

(l) If $e = u_{n-2} u_n$, then $\gcd(f(u_{n-2}), f(u_n)) = \gcd(4n-3, 4n-11) = 1$.

(m) If $e = u_{n-1} u_1$, then $\gcd(f(u_{n-1}), f(u_1)) = \gcd(4n-7, 1) = 1$.

(n) If $e = u_n u_2$, then $\gcd(f(u_n), f(u_2)) = \gcd(4n-11, 5) = 1$.

Thus, from each of the cases, *f* admits odd prime labeling on *P*(*n*, 2). Thus *P*(*n*, 2) is an odd prime graph.

An immediate conjecture that follows is:

Conjecture 2.13: Generalized Petersen graphs $P(n, k)$ are odd prime graphs.

It is evident from the paper that odd prime graphs need not be prime graphs. But what about the converse? This question gives us the following conjecture:

Conjecture 2.14: A prime graph is always an odd prime graph.

If this conjecture becomes true, many theorems proved in this paper will become trivial. Although it will still remain an interesting area to investigate about odd prime graphs that are not prime graphs.

3. Conclusion

A new labeling technique has been introduced in this paper called an odd prime labeling. Some standard graphs like path and complete bipartite graphs under certain conditions are proved as odd prime graphs. This paper also shows that wheel graph, several wheel related graphs and generalized Petersen graphs $P(n, 2)$ are odd prime graphs. Many graphs that do not admit prime labeling do admit odd prime labeling like a wheel graph when n is odd and the complete graph K_4.

4. Open Problems

There is a wide scope of investigating several graphs or graph families that are odd prime graphs but not prime graphs. Investigating $P(n, k)$ in general to be an odd prime graph can also become an interesting topic.

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