Generalized Results for Fixed Point Theorems in Complex Valued Metric Spaces

J. Uma Maheswari[1] & A. Anbarasan[2] & M. Ravichandran[2*]
[1.] Department of Mathematics, St. Joseph's College (Autonomous),
Tiruchirappalli - 620 002, India.
[2.] Department of Mathematics, Sri Sankara Arts and Science College(Autonomous),
Enathur, Kanchipuram-631561, India.

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ABSTRACT: *In this paper we generalized Results common fixed point theorems for rational type contraction mappings on complex valued metric spaces due to Fayaaz et.al., Azam et.al., and others.*

Key Words: *Fixed point; Common Fixed point ; Complex Valued metric spaces; Contractive type mappings.*

1 Introduction

Azam et al.[see[4]] in 2011 introduced the new concepts of complex valued metric spaces and find that the existence of contractive conditions in fixed point theorem for mappings satisfying rational type expressions.So many researchers are interested and generalized many concepts in complex valued metric spaces.

Sintunavarat et al.[see[3]] proved common fixed point results in complex valued metric spaces and also authors like Fayyaz et al. and others are interested proved results.Although , we are interested give some good results of Analysis involving in this division.

2 Preliminaries

We recall some basic notions and definitions which will be useful for proving our main results.

Let C be the set of Complex numbers and $z_1, z_2 \varepsilon$ C. Define partial order ≤ on C as follows.

$z_1 \leq z_2$ if and only if Re.(z_1)≤ Re.(z_2) also Im.(z_1)≤ Im.(z_2)

It follows that $z_1 \leq z_2$ if one of the following condition hold

1. $Re.\ z_1 = Re.\ z_2 \quad , Im.\ z_1 \prec Im.\ z_2$
2. $Re.\ z_1 \prec Re.\ z_2 \quad , Im.\ z_1 = Im.\ z_2$
3. $Re.\ z_1 \prec Re.\ z_2 \quad ,Im.\ z_1 \prec Im.\ z_2$
4. $Re.\ z_1 = Re.\ z_2 \quad , Im.\ z_1 = Im.\ z_2$

In particular, we will write $z_1 \prec z_2$ if $z_1 \neq z_2$ and one of (1), (2) and (3) is satisfied and we will write $z_1 \prec z_2$ if only (3) is satisfied.

Definition 2.1 Let X be a nonvoid set. A mapping $\rho : X \times X \to C$ is called a complete valued metric space on X if the following conditions are satisfied

(1)$0 \leq \rho(x,y)$ for all $x, y \in X$ and $\rho(x,y) = 0$ if and if x=y.

(2)$\rho(x,y) = \rho(y,x)$ for all $x, y \in X$

(3)$\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ for all $x, y \in X$.

Definition 2.2 Let (X, ρ) be a complete valued metric space

(1)Let $\{x_n\}$ be a Cauchy sequence if for every 0<c ε C find a integer N such that $\rho(x_n, x_m) < c$ for every m,n ≥ N.

(2)Let $\{x_n\}$ converges to an element xε X if for every 0<c ε C find a intger N such that $\rho(x_n, x) < c$ for all n≥N.

(3)Suppose that (X, ρ) is complete if for every Cauchy sequence in X converge to a point in X.

Lemma 2.1 Let (X,ρ) be a complex valued metric space and let $\{x_n\}$ be a sequence in X.Then $\{x_n\}$ converges to x if and only if $|\rho(x_n,x)| \to 0$ as $n \to \infty$.

Lemma 2.2 Let (X,ρ) be a complex valued metric space and let $\{x_n\}$ be a sequence in X.Then $\{x_n\}$ Cauchy sequence if and only if $|\rho(x_n,x_{n+m})| \to 0$ as $n \to \infty$.

3 Main Results

In this section we generalized (see[5]) and proved our main results.

Theorem 3.1 Let K and L be self mappings defined on a complete complex valued metric space (X,ρ) satisfying the condition that

$$\rho(Kx,Ly) \le \alpha\rho(x,y) + \frac{\beta\rho(x,Ly)\rho(y,Kx)\rho(Kx,Ly)}{1+(\rho(x,y))^2} + \frac{\gamma\rho(x,Kx)\rho(y,Ly)\rho(Kx,Ly)}{1+\rho(x,y)\rho(Kx,Ly)}$$ for every

x, yε X where α,β,γ are non negative real's with $\alpha+\beta+\gamma<1$ then K and L have a unique common fixed point.

Proof Let us consider $x_0 \in X$.

Define a $x_{2m+1} = Kx_{2m}$, $x_{2m+2} = Lx_{2m+1}$ m=0,1,2,….. then

$$\rho(Kx_{2m},Lx_{2m+1}) \le \alpha\rho(x_{2m},x_{2m+1}) + \frac{\beta\rho(x_{2m},Lx_{2m+1})\rho(x_{2m+1},Kx_{2m})\rho(Kx_{2m},Lx_{2m+1})}{1+(\rho(x_{2m},x_{2m+1}))^2} +$$

$$\frac{\gamma\rho(x_{2m},Kx_{2m})\rho(x_{2m+1},Lx_{2m+1})\rho(Kx_{2m},Lx_{2m+1})}{1+\rho(x_{2m},x_{2m+1})\rho(Kx_{2m},Lx_{2m+1})}$$

Since $x_{2m+1} = Kx_{2m}$ therefore $\rho(x_{2m+1},Kx_{2m}) = 0$

Now,

$$\rho(x_{2m+1},x_{2m+2}) \le \alpha\rho(x_{2m},x_{2m+1}) + \frac{\gamma\rho(x_{2m},x_{2m+1})(\rho(x_{2m+1},x_{2m+2}))^2}{1+\rho(x_{2m},x_{2m+1})\rho(x_{2m+1},x_{2m+2})}$$

$$|\rho(x_{2m+1},x_{2m+2})| \le \alpha|\rho(x_{2m},x_{2m+1})| + \frac{\gamma|\rho(x_{2m},x_{2m+1})||(\rho(x_{2m+1},x_{2m+2}))^2|}{|1+\rho(x_{2m},x_{2m+1})\rho(x_{2m+1},x_{2m+2})|}$$

Since $|1+\rho(x_{2m},x_{2m+1})\rho(x_{2m+1},x_{2m+2})| > |\rho(x_{2m},x_{2m+1})\rho(x_{2m+1},x_{2m+2})|$

So that,

$$|\rho(x_{2m+1},x_{2m+2})| \le \alpha|\rho(x_{2m},x_{2m+1})| + \gamma|\rho(x_{2m+1},x_{2m+2})|$$

$$|\rho(x_{2m+1},x_{2m+2})| \le \frac{\alpha}{1-\gamma}|\rho(x_{2m},x_{2m+1})|$$

Letting $g = \dfrac{\alpha}{1-\gamma}$,

We have, $|\rho(x_s,x_{s+1})| \le g|\rho(x_{s-1},x_s)| \le g^2|\rho(x_{s-2},x_{s-1})| \le \ldots\ldots \le g^n|\rho(x_0,x_1)|$

Then for any t>s, we obtain

$$|\rho(x_s,x_t)| \le |\rho(x_s,x_{s+1})| + |\rho(x_{s+1},x_{s+2})| + \ldots\ldots + |\rho(x_{t-1},x_t)|$$

$$|\rho(x_s,x_t)| \le [g^s + g^{s+1} + \ldots\ldots + g^{t-1}]|\rho(x_0,x_1)|$$

Simplifying, we obtain that

$$\left|\rho(x_s,x_t)\right| \le [\frac{g^n}{1-g}]\left|\rho(x_0,x_1)\right|$$

Let as $s \to \infty$ we get

$$\left|\rho(x_s,x_t)\right| \to 0$$

By lemma 2.2 says the sequence $\{x_s\}$ is a Cauchy sequence. Since X is complete, we find that some hεX such that $x_s \to h$ as $s \to \infty$.

Suppose on the contrary that, let $h \ne Kh$.Let us consider $\rho(h,Kh)=u>0$

Therefore, we have

$$u=\rho(h,Kh) \le \rho(h,Lx_{2m+1})+\rho(Lx_{2m+1},Kh)$$
$$\le \rho(h,x_{2m+2})+\alpha\rho(h,x_{2m+1})+\frac{\beta\rho(h,Lx_{2m+1})\rho(x_{2m+1},Kh)\rho(Kh,Lx_{2m+1})}{1+(\rho(h,x_{2m+1}))^2}+$$
$$\frac{\gamma\rho(h,Kh)\rho(x_{2m+1},Lx_{2m+1})\rho(Kh,Lx_{2m+1})}{1+\rho(h,x_{2m+1})\rho(Kh,Lx_{2m+1})}$$

Then we obtain for all m,

$$\left|\rho(h,Kh)\right| \le \left|\rho(h,x_{2m+2})\right|+\alpha\left|\rho(h,x_{2m+1})\right|+\frac{\beta\left|\rho(h,x_{2m+2})\right|\left|\rho(x_{2m+1},Kh)\right|\left|\rho(Kh,x_{2m+2})\right|}{\left|1+(\rho(h,x_{2m+1}))^2\right|}+$$
$$\frac{\gamma\left|u\right|\left|\rho(x_{2m+1},x_{2m+2})\right|\left|\rho(Kh,x_{2m+2})\right|}{\left|1+\rho(h,x_{2m+1})\rho(Kh,x_{2m+2})\right|}$$

Letting $s \to \infty$, we get ,

$\rho(h,Kh)=0$ which is contradiction that of our choice.

Therefore, h=Kh.

Similarly, we can also prove that h=Lh.

To prove the existence of uniqueness of common fixed point

Let us consider h* be in X which is the another common fixed point of K and L

i.e. h* =K h*=T h*

Then, $\rho(h,h^*)=\rho(Kh,Lh^*) \le \alpha\rho(h,h^*)+\frac{\beta\rho(h,h^*)\rho(h^*,h)\rho(h,h^*)}{1+(\rho(h,h^*))^2}+\frac{\gamma\rho(h,h)\rho(h^*,h^*)\rho(h,h^*)}{1+\rho(h,h^*)\rho(h,h^*)}$

$$=\alpha\rho(h,h^*)+\frac{\beta\rho(h,h^*)\rho(h^*,h)\rho(h,h^*)}{1+\rho(h,h^*)}$$

So that , $\left|\rho(h,h^*)\right| \le \alpha\left|\rho(h,h^*)\right|+\frac{\beta\left|\rho(h,h^*)\right|\left|\rho(h^*,h)\right|\left|\rho(h,h^*)\right|}{1+(\rho(h,h^*))^2}$

Therefore , $\left|\rho(h,h^*)\right| \le (\alpha+\beta)\left|\rho(h,h^*)\right|$

Since, $\left|1+(\rho(h,h^*))^2\right| > \left|(\rho(h,h^*))^2\right|$

Which implies , $\left|\rho(h,h^*)\right| < \left|\rho(h,h^*)\right|$ since $\alpha+\beta<1$

Which is contradiction to our choice , so that h=h*

Hence, Uniqueness existence of common fixed point.

By putting K=L, then the theorem3.1 can be viewed as an extension of Bryant theorem to complex valued metric spaces.

Corollary 3.2 Let $L: X \to X$ be a self mapping defined on a complete complex valued metric space (X,ρ) satisfying the condition $\rho(L^n x, L^n y) \le \alpha\rho(x,y)$ for all x, yεX where α is non negative real with α<1 then L is one and only one point x in X such that Lx=x.

Example 3.3 Let X=C be the set of complex numbers. Define $f: C\times C \to C$ as follows where $z_1=x_1+iy_1$, $z_2=x_2+iy_2$.Then (C, f) is a complete complex valued metric space.
Define $L: C \to C$ as

$$L(x) = \begin{cases} 0 \quad if \quad x, y \in Q \\ 3+2i \quad if \quad x, y \in Q^c \\ 3 \quad if \quad x \in Q^c, y \in Q \\ 4i \quad if \quad x \in Q, y \in Q^c \end{cases}$$

Let us consider $x=\sqrt{3}$ and $y=0$ we obtain

$$f(L(\sqrt{3}), L(0)) = f(3,0) = 3 \le \alpha f(\sqrt{3},0) = \alpha\sqrt{3}$$

Therefore, $\alpha \ge \sqrt{3}$ which is contradiction as $0 \le \alpha < 1$.

We notice that L^2 z=0 so that $0 = f(L^2 z_1, L^2 z_2) \le \alpha f(z_1, z_2)$ which shows that L^2 satisfies the requirement of Bryant theorem and z=0 is the unique fixed point of T.

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