THE LINE MYCIELSKIAN GRAPH OF A GRAPH

Keerthi G. Mirajkar[1] & Veena N. Mathad[2]
[1]Department of Mathematics, Karnatak University's Karnatak Arts College,
Dharwad - 580001, Karnataka, India
[2]Department of Mathematics, University of Mysore, Manasagangotri, Mysore-06, India

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ABSTRACT: *In this paper, we introduce the concept of the line mycielskian graph of a graph. We obtain some properties of this graph. Further we characterize those graphs whose line mycielskian graph and mycielskian graph are isomorphic. Also, we establish characterization for line mycielskian graphs to be eulerian and hamiltonian.*
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I. Introduction

By a graph $G = (V,E)$ we mean a finite, undirected graph without loops or multiple lines. For graph theoretic terminology, we refer to Harary [3].

For a graph G, let $V(G)$, $E(G)$ and $L(G)$ denote the point set, line set and line graph of G, respectively.

The line degree of a line uv of a graph G is the sum of the degree of u and v. The open-neighborhood$N(u)$ of a point u in $V(G)$ is the set of points adjacent to u. For each u_i of G, a new point u'_i is introduced and the resulting set of points is denoted by $V_1(G)$.

Mycielskian graph$\mu(G)$ of a graph G is defined as the graph having point set $V(G) \cup V_1(G) \cup v$ and line set $E(G) \cup \{xy' : xy \in E(G)\} \cup \{y'v : y' \in V_1(G)\}$.

Motivated by this concept on the same lines we introduce the concept of line mycielskian graph of a graph as follows.

The open-neighborhood$N(e_i)$ of a line e_i in $E(G)$ is the set of lines adjacent to e_i.

For each line e_i of G, a new point e'_i is taken and the resulting set of points is denoted by $E_1(G)$.

The line mycielskian graph $L\mu(G)$ of a graph G is defined as the graph having point set $E(G) \cup E_1(G) \cup \{e\}$ and the line set $E(L(G)) \cup \{e_ie'_j : e_i \cdot e_j|$are adjacent lines in $G\} \cup \{e'_je : e'_j \in E_1(G)\}$.

In Figure 1, a graph G, μ(G) and $L\mu(G)$ are shown.

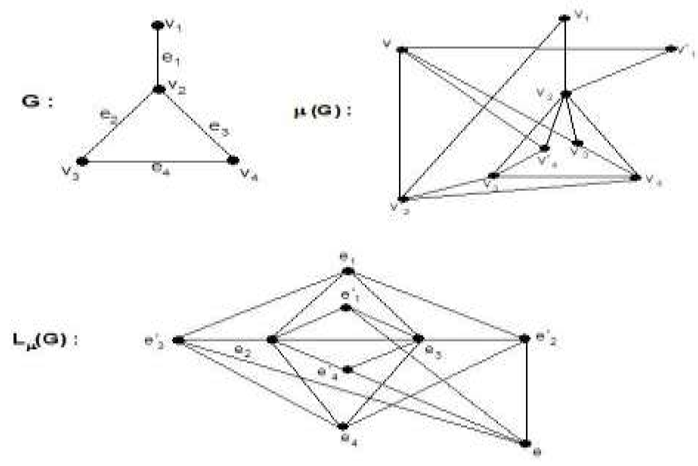

Figure 1.

The graph G is an induced subgraph ofμ(G). The line graph $L(G)$ is an induced subgraph of $L\mu(G)$. To present the complete results, some important theorems which are used through out the paper are mentioned below.

Remark 1. If $G = L\mu(H)$ for some graph H, then $G = \mu(L(H))$:

Remark 2. A mycielskian graph $\mu(G)$ is a unicyclicgraph if and only if $G = P_2$:

Remark 3. Let $e_j \in E(G)$. If the neighbours of e_j in G are $e_1, e_2, \ldots, e_k$ then the neighbours of e_j in $L\mu(G)$ are $e_1, e_2, \ldots, e_k, e'_1, e'_2, \ldots, e'_k$. Hence $deg_{L_\mu(G)}e_j = 2deg_{L(G)}e_j$, $deg_{L_\mu(G)}e'_j = deg_{L(G)}e_j + 1$, $deg_{L_\mu(G)}e = |E(G)|$, where e is the point in $L\mu(G)$ adjacent to each e'_i.

Theorem A [3]. G is eulerian if and only if every point of G is of even degree.

Theorem B [3]. If G is a (p, q) graph whose points have degree d_i then $L(G)$ has q points and q_L lines, where $q_L = -q + \frac{1}{2}\sum_{i=1}^{p} {d_i}^2$.

Theorem C [2]. Let G be a graph without isolated points. Then $L(G)$ is hamiltonian if and only if $G = K_{1,l}$ for some $l \geq 3$, or G contains a dominating circuit.

II. Results

Theorem 2.1. For any line in a graph G, with line degree n, the degree of the corresponding point in $L\mu$ is $2(n - 2)$.

Proof. If a line e in G is of line degree n, then e is adjacent to $n - 2$ lines, say $e_1, e_2, \ldots, e_{n-2}$. Since the line itself contributes one degree to each line adjacent to it in G, $(n - 2)$ degrees are contributed for the corresponding point in $L\mu(G)$. Hence $deg_{L_\mu(G)}e_j = 2(n - 2)$.

Theorem 2.2. If G is a nontrivial connected (p, q) graph, whose points have degree d_i and j_i is the number of lines adjacent to the line e_i in G, then the line mycielskian graph $L\mu(G)$ has $2q + 1$ points and $\frac{1}{2}\sum_{i=1}^{p} {d_i}^2 + \sum_{i=1}^{q} {j_i}^2$ lines.

Proof. By the definition of line mycielskian graph, $L\mu(G)$ has $2q + 1$ points. $L\mu(G)$ constitutes the line graph $L(G)$ as an induced subgraph and by Theorem B, $L(G)$ has $-q + \frac{1}{2}\sum_{i=1}^{p} {d_i}^2$ lines. Also, each e'_i of $E_1(G)$ produce $j_i + 1$ lines in $L\mu(G)$, $1 \leq i \leq q$. So the number of lines in $L\mu(G)$ is the sum of the number of lines in $L(G)$ and the number of lines adjacent to each e'_i in $L\mu(G)$. Hence the number of lines in $L\mu(G)$ is

$$= -q + \frac{1}{2}\sum_{i=1}^{p} d_i^2 + (-q) + \sum_{i=1}^{q}(j_i + 1)$$

$$= \frac{1}{2}\sum_{i=1}^{p} d_i^2 + \sum_{i=1}^{q} j_i$$

Theorem 2.3. Let G be a nontrivial connected (p, q) graph. The $\mu(G)$ is isomorphic to $L\mu(G)$ if and only if G is a cycle.

Proof. Suppose $\mu(G)$ is isomorphic to $L\mu(G)$. By definitions of $\mu(G)$ and $L\mu(G)$, G has equal number of points and lines. So G is unicyclic. Now, suppose G is unicyclic other than a cycle, then $G \ncong L(G)$ so that $\mu(G)$ is not isomorphic to $L\mu(G)$, a contradiction. Hence G is a cycle.

Conversely, suppose G is a cycle. Since the number of points and number of lines in a cycle are equal and $G \ncong L(G)$, it follows that $\mu(G)$ is isomorphic to $L\mu(G)$.

Theorem 2.4. A graph G is line mycielskian graph, if and only if the point set of G can be partitioned into three subsets V_1, V_2 and V_3 such that there exists a bijection $f: V_1 \rightarrow V_2$ with $N(f(u)) - V_3 = N(u) \cap V_1$, for all $u \in V_1$.

Proof. Let $G = L(H)$ for some graph H. Then $G = \mu(L(H))$. To construct G from H, we add a new point for each line e_i of H and make it adjacent to all points of $L(H)$, which correspond to the lines of H adjacent to e. We take one more point e and make it adjacent to each e'_i. Let $V_1 = V(L(H)) = E(H)$, $V_3 = e$ and $V_2 = V(G) - V(L(H)) - V_3 = V(G) - E(H) - V_3$. Let $v_1 \in V_1$ and $v_2 \in V_2$. Then $v_1 \rightarrow v_2$ defines bijection of V_1 onto V_2 and $N(v_2) - V_3 = N(v_1) \cap V_1$.

Conversely, suppose $V(G) = V_1 \cup V_2 \cup V_3$ is a partition with $f: V_1 \rightarrow V_2$ a bijection such that $N(f(u)) - V_3 = N(u) \cap V_1, \forall u \in V_1$. By associating different members of V_2 with different members of V_1, it follows that $G = L\,\mu(H)$, where $L(H) = \langle V_1 \rangle$.

Theorem 2.5. The line mycielskian graph $L\mu(G)$ of a graph G is unicyclic if and only if $G = P_3$.

Proof. Suppose $L\mu(G)$ is unicyclic. Then $\mu(L(G))$ is unicyclic. By Remark 2, $L(G)$ is P_2 and hence $G = P_3$. Conversely, if $G = P_3$, then $L\mu(G) = C_5$ the unicyclic graph.

The **girth** of a graph G, denoted by $g(G)$, is the length of a shortest cycle (if any) in G [3]. In the following theorem, we obtain the girth of a line mycielskian graph.

Theorem 2.6. For any connected graph G of order atleast three,

$$g(L_\mu(G)) = \begin{cases} 3 & \text{if } K_3 \text{ or } K_{1,3} \text{ is subgraph of G,} \\ 5 & G = P_3, \\ 4 & \text{otherwise.} \end{cases}$$

Proof. Let G be a connected graph of order $p \geq 3$. If either K_3 or $K_{1,3}$ is a subgraph of G then $L\mu(G)$ contains a triangle and hence its girth is 3. If $G = P_3$, then by Theorem 5, $L\mu(G) = C_5$ and hence its girth is 5. Otherwise G contains P_4 as a subgraph and the graph given in Figure 2 is a subgraph of $L\mu(G)$. Hence $g(L\mu(G)) = 4$.

Theorem 2.7. The line mycielskian graph $L\mu(G)$ is eulerian if and only if the following conditions hold.
(i) G has even number of lines.
(ii) Every line in G is of odd line degree.

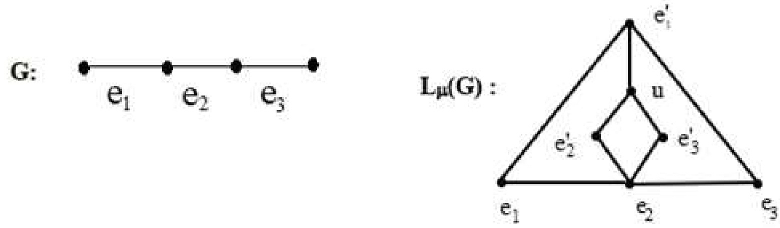

Figure 2.

Proof. Suppose $L\mu(G)$ is eulerian. Then by Theorem A, every point of $L\mu(G)$ is of even degree. We consider the following cases.
We consider the following cases,

Case 1. Suppose G has odd number of lines. Then by Remark 3, *degeis* odd. By Theorem A, a contradiction. Hence G must contain even number of lines.

Case 2. Suppose G has a line e_i with even line degree. Then by Remark 3, $deg_{L_\mu(G)}e_j = 2deg_{L(G)}e_j$ is even. But $deg_{L_\mu(G)}e'_i = deg_{L(G)}e_i + 1$, is odd. By Theorem A, $L\mu(G)$ is not eulerian, a contradiction. Thus every line must be of odd line degree.

Conversely, suppose (i) and (ii) hold. Then it is easy to see by Remark 2, that each point of $L\mu(G)$ is of even degree. Hence $L\mu(G)$ is eulerian.

A set S of lines of a graph G is a **dominating set** of lines if every line of G either belongs to S or is adjacent to an element of S. If $<S>$ is a circuit C, then C is called a **dominating circuit** of G. Equivalently, a circuit C in a graph G is a dominating circuit if every line of G is incident with a point of C.

Theorem 2.8. Let G be a nontrivial connected (p,q) graph. Then $L\mu(G)$ is hamiltonian if and only If
(i) $G = K_{1,l}$, $l \geq 2$ or
(ii) G contains a dominating circuit.

Proof. We consider the following cases.

Case 1. If $G = K_{1,l}$, $l \geq 3$, or G has a dominating circuit, then by Theorem C, $L(G)$ is Hamiltonian. Let $(e_1, e_2, \ldots, e_n, e_1)$ be a hamiltonian cycle in $L(G)$. Let $\{e'_1, e'_2, \ldots, e'_n\}$ be the set of newly introduced points in $L\mu(G)$ for each e_i, $1 \leq i \leq n$ and e be the point in $L\mu(G)$ adjacent to $e'_1, e'_2, \ldots, e'_n$. Then $(e_1, e'_2, e_3, e'_4, \ldots, e_{n-2}, e'_{n-1}, e, e'_n, e_{n-1}, e'_{n-2}, e_{n-3}, \ldots, e'_5, e_4, e'_3, e_2, e'_1, e_n, e_1)$ is a hamiltonian cycle in $L\mu(G)$.

Case 2. If $G = K_{1,2}$, then $L\mu(G) = C_5$, which is hamiltonian. Hence, in both cases, $L\mu(G)$ is hamiltonian.

Conversely, suppose $L\mu(G)$ is hamiltonian. Consider the hamiltonain cycle Z in $L\mu(G)$. Let e_i, e'_i,$1 \leq i \leq q$ and e be the points of $L\mu(G)$, where e_i's are the points of $L(G)$, e_i's are the newly introduced points of $L\mu(G)$ corresponding to each e_i, $1 \leq i \leq q$. Since $\{e'_i ; 1 \leq i \leq q\}$is an independent set and $e \in N(e'_i)$ for each $1 \leq i \leq q$, it follows that any hamiltonian of $L\mu(G)$ contains e_i's as alternate points except at one position.

We consider the following cases.

Case 1.$q>p$. Let $Z = (e_1, e'_2, e_3, e'_4, \ldots, e_{q-2}, e, e'_q, e_{q-1}, e'_{q-2}, \ldots, e_4, e'_3, e_2, e'_1, e_q, e_1)$ for odd q and $(e_1, e'_2, e_3, e'_4, \ldots, e_{q-1}, e'_q, e, e'_1, e_2, e'_3, e_4, \ldots, e_{q-2}, e'_{q-1}, e_q, e_1)$ for even q.

In both cases, a hamiltonian cycle Z_1 of $L(G)$ can be obtained from Z by deleting e'_i; $1 \leq i \leq q$ and e. i.e. $Z_1 = (e_1, e_2, \ldots, e_{q-1}, e_q, e_1)$ is a hamiltonian cycle of $L(G)$. By Theorem C, it follows that $G = K_{1,q}$, for some $q \geq 3$ or G contains a dominating circuit.

Case 2. Suppose $q = 2$ which implies $G = k_{1;2}$ and $L\mu(G) = C_5$, a hamiltonian graph.

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