A NOTE ON UNIT GRAPH OF A RING

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ABSTRACT: *The zero divisor graph of a commutative ring R was introducedby Beck [1], and later studied by Vasanthakandasamy [24]. In this short note, we introduced the concept namely "Unit graph of type-1" denotedby $UG1(R)$in Associative rings R and announced few important fundamental results. We proved that $UG1(Z_n)$ contains atleast one vertex of degree 1 and if F is a field with $|F|$ is odd, then $UG1(F)$ contains at least one vertex of degree 1. We included some examples.*

Key Words*: Graph, Unit graph of a ring, Star graph*

Mathematics Subject Classification*: 05C07, 05C20, 05C76, 05C99, 13E15*

1. 1Introduction

Let G = (V, E) be a graph consist of a finite non-empty set V of vertices and finite set E of edges such that each edge e_k is identified as an unordered pair of vertices $\{v_i, v_j\}$, where v_i, v_j are called end points of e_k. The edge $e_k$is also denoted by either $v_i v_j$ or $\overline{v_i v_j}$. We also write $G(V,E)$ for the graph. Vertex set and edge set of G are also denoted by $V(G)$ and $E(G)$ respectively. An edge associated with a vertex pair $\{v_i, v_i\}$ is called a self-loop. The number of edges associated with the vertex is the degree of the vertex, and d(v) denotes the degree of the vertex v. If there is more than one edge associated with a given pair of vertices, then these edges are called parallel edges or multiple edges. A graph that does not have self-loops or parallel edges is called a simple graph. We consider simple graphs only.

1.2 Definitions: (i) A graph G (V, E) is said to be a **star graph** if there exists a fixed vertex v (called the center of the star graph)such that E = {vu / u $\in$ V and u $\neq$ v}. A star graph is said to be an **n-star graph** if the number of vertices of the graph is n.

(ii) In a graph G, a subset S of V(G) is said to be a ***dominating set*** if every vertex not in S has a neighbour in S. The ***domination number***, denoted by γ(G) is defined as min {|S| / S is a dominating set in G}.

(iii**)** (Vasanthakandasamy and FlorentinSmarandache [24])A graph $G=(V,E)$ is said to be the **zero divisor graph** of a commutative ring R if $V=R$ and

E = $\left\{\overline{xy} / x\neq y,\ x,y\in R,\ x\neq 0\neq y,\ xy=0\right\}\cup\left\{\overline{x0} / 0\neq x\in R\right\}$ where $\overline{xy}$ denotes an edge between $x, y\in V$.

This definition 'zero divisor graph' is same as that of Beck [1988].

1.3 Notation: (i) We denote zero divisor graph of ring R by ZDG(R)

(ii) In the graph ZDG(R), we have that $V(ZDG(R))=R$ and

$E(ZDG(R))=\left\{\overline{xy} / x\neq y,\ x,y\in R, x\neq 0\neq y,\ xy=0\right\}\cup\left\{\overline{x0} / 0\neq x\in R\right\}$

1.4 Examples(i): (Vasanthakandasamy and FlorentinSmarandache [24]) ConsiderZDG(R) with $R=\mathbb{Z}_8$. We knowthatR $=\mathbb{Z}_8=\{0,1,2,3,4,5,6,7\}$. SoV(ZDG(R)) $=\{0,1,2,3,4,5,6,7\}$. Since $2.4=4.6=0$ (mod 8), there exist edges between the vertices

2 and 4, 4 and 6. Since '0' is adjacent to all the elements in R, we get$\overline{01}, \overline{02}, \overline{03}, \overline{04}, \overline{05}, \overline{06}, \overline{07}\in$ E(ZDG(R)).

Therefore,$E(ZDG(R)) = \{\overline{01}, \overline{02}, \overline{03}, \overline{04}, \overline{05}, \overline{06}, \overline{07}, \overline{24}, \overline{46}\}$.Now ZDG(R) is given in Figure 1.4(i).We observe that ZDG (Z_8) contains 8-star graph as its subgraph.
he domination number is 1.

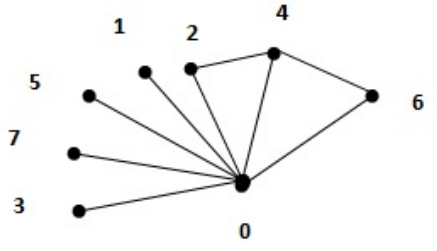

Figure-1.4(i)

(ii) Consider ZDG(R) with$R = \mathbb{Z}_9$. We know that$R = \mathbb{Z}_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.So $V(ZDG(R)) = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Since$3.6 = 0 \pmod 9$, there exist edges between the vertices 3 and 6. Since '0' is adjacent to all the elements in R, we get$\overline{01}, \overline{02}, \overline{03}, \overline{04}, \overline{05}, \overline{06}, \overline{07}, \overline{08} \in E(ZDG(R))$.
Therefore,$E(ZDG(R)) = \{\overline{01}, \overline{02}, \overline{03}, \overline{04}, \overline{05}, \overline{06}, \overline{07}, \overline{08}, \overline{36}\}$. Now ZDG(R) is given in Figure 1.4(ii). We observe that ZDG (Z_9) contains 9-star graph as its subgraph. The domination number is 1.

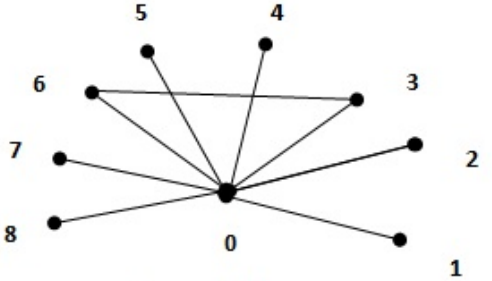

Figure-1.4(ii)

UNIT GRAPH OF TYPE-1

2.1 DEFINITION: Let R be an associative ring with 1. Write U(R) the set of all units in R. A graph $G = (V, E)$ is said to be the unit graph of type-1 of R if$V =$ Figure-1.4(ii) $/ 1 \neq x, x \in U(R)\} \cup \{\overline{xy} / x, y \in U(R), x \neq 1 \neq y$ and $xy = 1\}$.
If R is a commutative ring, then this definition of 'Unit graph of type-1' coincide with the definition of'Unit graph' mentioned by Vasanthakandaswamy(2009).

2.2 Notation: (i)The unit graph of type-1 of a ring R is denoted by UG1(R).
(ii) In the graph UG1(R) we have that$V(UG1(R)) = U(R)$and
$E(UG1)=\{\overline{1x} / 1 \neq x, x \in U(R)\} \cup \{\overline{xy} / x, y \in U(R), x \neq 1 \neq y$ and $xy = 1\}$

2.3 Example: Let us construct the graph UG1(R), where$R = Z_8$,$U(R) = \{1, 3, 5, 7\}$. So$V(UG1(R)) = \{1, 3, 5, 7\}$. Since '1' is adjacent to all units in R, we have$\overline{13}, \overline{15}, \overline{17}$. There areno other edges because $\{\overline{xy} / x, y \in U(R), x \neq 1 \neq y$ and $xy = 1\} = \emptyset$. So $E(UG1) = \{\overline{13}, \overline{15}, \overline{17}\}$ and the graph UG1(R)is given in the Fig. 2.3.

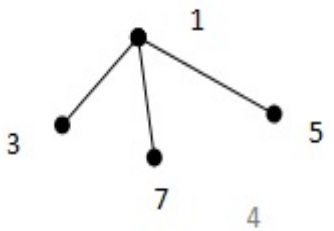

Fig. 2.3

2.4 Example: Let us construct the graph UG1(R), where$R = Z_9$. Here $U(R) = \{1, 2, 4, 5, 7, 8\}$and $V(UG1(R)) = \{1, 2, 4, 5, 7, 8\}$.Since '1' is adjacent to all units in R, we have the edges$\overline{12}, \overline{14}, \overline{15}, \overline{17}, \overline{18}$. Since $2.5 = 4.7 = 1$, there is an edge between 2 and 5, 4 and 7,so $\{\overline{xy}/\ x, y \in U(R),\ x \neq 1 \neq y \text{ and } xy = 1\} = \{\overline{25}, \overline{47}\}$.Therefore, $E(UG1(R)) = \{\overline{12}, \overline{14}, \overline{15}, \overline{17}, \overline{18}, \overline{25}, \overline{47}\}$.Now UG1(R)is given in Fig. 2.4

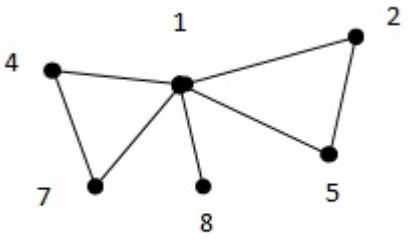

Fig. 2.4

2.5 Result: (i) For any ring R, both the graphs ZDG(R) and UG1(R) are simple graphs
(ii) There are no common edges inZDG(R) and UG1(R)

Proof: (i) is clear

(ii) We have to verifythat $E(ZDG(R)) \cap E(UG1(R)) = \emptyset$.

In a contrary way, suppose that$\overline{xy} \in E(ZDG(R)) \cap E(UG1(R))$. Now$\overline{xy} \in E(UG1(R)), x, y \in U(R)$, $x \neq 0$and $y \neq 0$. Since $y \in U(R)$ there exists $0 \neq z \in R$ such that $yz = 1$.
Since $\overline{xy} \in E(ZDG(R))$, we have that$xy = 0, x \neq 0, y \neq 0$
Now $0 = 0.z = (xy)z = x(yz) = x.1 = x$,a contradiction.
Hence$E(ZDG(R)) \cap E(UG1(R)) = \emptyset$.
The proof is complete.

2.6 Note: Examples1.4(i), 1.4(ii), 2.3&2.4 provides two illustrations for $E(ZDG(R)) \cap E(UG1(R)) = \emptyset$.

2.7Result: $UG1(Z_n)$ contains atleast one Vertex of degree 1.

Proof: Consider Z_n.
Since$(n-1)^2 = n^2 - 2n + 1 \equiv 1(\text{mod } n)$, we have that $(n-1)$ is a self invertible element. By definition of UG1(R), the vertex $(n-1)$ and the vertex 1 are connected by an edge; and there is no other vertex adjacent to $(n-1)$. Hence the degree of the vertex $(n-1)$ is 1.

2.8 Result: If F is a field with $|F|$is odd, then $UG1(F)$ contains at least one vertex of degree 1.

Proof: Suppose $|F|$ odd. Then $F \setminus \{0\}$ is a multiplicative group with $|F \setminus \{0\}|$ is even. Since all the elements of$F \setminus \{0\}$ are units we get that $V(UG1(F)) = F \setminus \{0\}$. Also $1 \in F \setminus \{0\}$and so$|F \setminus \{0, 1\}|$ is odd.
We wish to verify that there is at least one vertex which is selfinvertible. In a contrary way, suppose there is no element in $F \setminus \{0, 1\}$ which is self invertible. Then take $x_1 \in F \setminus \{0, 1\}$ and its inverse x_1^{-1} and then pair them$\{x_1, x_1^{-1}\}$. By our supposition for all elements the corresponding inverses exists. If there are k such pairs formed then $|F \setminus \{0, 1\}| = 2.k$ (an even number), a contradiction.
The proof is complete.

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