EVOCATIVE PROOFS OF FIXED POINT THEOREMS IN METRIC SPACE

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ABSTRACT: *In this article elaborate proofs of Fixed point theorems are presented by applying the infimum property of real numbers without linking the traditional iterative procedure. An interesting consequence of such application is that the obtained fixed point can be shown as a contractive fixed point. Even though these proofs are analytical rather than fixative, their appeal lies in their innocuous handling and presentation.*

Key Words: *The Infimum Property, Metric Space, Self-map, Fixed Point, Contractive Fixed Point.*
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1. INTRODUCTION

Let Y be a nonempty set and $\rho : Y \times Y \to \square$ such that

(d1) $\rho(a,b) \geq 0$ for all $a,b \in Y$ with $\rho(x,y) = 0$ iff $a = b$,

(d2) $\rho(a,b) = \rho(b,a)$ for all $a,b \in Y$

(d3) $\rho(x,y) \leq \rho(x,y) + \rho(y,z)$ for all $a,b,c \in Y$

Then the pair (Y,ρ) is called a Metric space with Metric ρ on Y. Axioms (d2) and (d3) are referred to as the *symmetry* and the *triangle inequality* respectively. The notion of Metric space was introduced by Frechet [2] in 1907.

Definition 1.1. A sequence $\langle y_n \rangle_{n=1}^{\infty}$ in a Metric space (Y,ρ) is said to be convergent with limit $z \in X$ if for every $\grave{o} > 0$ there is a positive integer N such that $\rho(y_n, z) < \grave{o}$ for all $n \geq N$.

Definition 1.2. A sequence $\langle y_n \rangle_{n=1}^{\infty}$ in a Metric space (Y,ρ) is said to be Cauchy if for every $\grave{o} > 0$ there is a positive integer N such that $\rho(y_n, y_m) < \grave{o}$ for all $m,n \geq N$.

Note that every convergent sequence in a metric space (Y,ρ) is Cauchy.

Definition 1.3. A Metric space (Y,ρ) is said to be complete if every Cauchy sequence in X converges in it.

2. MAIN RESULTS

We begin this section with an important tool the infimum property of real numbers, as stated below:

Lemma 2.1. Let $D \subset \square$ be nonempty subset of real numbers which is bounded below. Then $k = \inf D$ exists.

An immediate consequence of Lemma 2.1 is:

Lemma 2.2. Let k be the infimum of $D \subset \square$. Then there exists a sequence $\langle y_n \rangle_{n=1}^{\infty}$ in D with $\lim_{n\to\infty} y_n = k$.

Definition 2.1. (Phaneendra et al, [3])

A fixed point z of T on a Metric space (Y,ρ) is a contractive fixed point of it if the orbital sequence $O_T(y) = \{y, fy, \ldots, f^n y, \ldots\}$ at each $y \in Y$ converges to z.

Example 2.1. Define $T : \square \to \square$ as $Ty = \frac{y}{3}$ for all $y \in Y$. Then we see that 0 is the unique fixed point of

T and for each $y \in Y$, the T-orbit $O_T(y) = \left\{ y, \frac{y}{3}, \frac{y}{3^2}, \ldots, \frac{y}{3^n}, \ldots \right\}$ converges to 0. That is, 0 is a Contractive fixed point of T.

Theorem 2.1. *Suppose that* (Y, ρ) *is a complete metric space and T, a self-map on Y satisfying the following contraction condition such that*

$$\rho(Ta,Tb) \le \alpha\rho(a,b) + \beta\rho(a,Ta) + \gamma\rho(b,Tb) + \delta \max\{\rho(a,Tb), \rho(b,Ta), \rho(a,Ta), \rho(b,Tb)\} \quad (2.1)$$

$$\textit{for all } a,b \in Y,$$

where $\alpha, \beta, \gamma, \delta \ge 0$ with $\alpha + \beta + \gamma + 2\delta < 1$. *Then* T *has a unique fixed point z in* Y.

Proof. Let $D = \{\rho(a,Ta) : a \in Y\}$.

Note that D is a nonempty set of nonnegative numbers which is bounded below. Hence by Lemma 2.1, it has the infimum, say $l \ge 0$.

We claim that $l = 0$.

If possible, we suppose that $l > 0$. Now from (2.1) with $b = Ta$ and the trianangle inequality (d3), we have

$$\rho(Ta,T^2a) \le \alpha\rho(a,Ta) + \beta\rho(a,Ta) + \gamma\rho(Ta,T^2a) + \delta \max\left\{\rho(a,T^2a), \rho(Ta,Ta), \rho(a,Ta), \rho(Ta,T^2a)\right\}$$
$$\le (\alpha+\beta)\rho(a,Ta) + \gamma\rho(Ta,T^2a) + \delta \max\left\{\rho(a,Ta) + \rho(Ta,T^2a), \rho(a,Ta), \rho(Ta,T^2a)\right\}$$
$$\le (\alpha+\beta)\rho(a,Ta) + \gamma\rho(Ta,T^2a) + \delta[\rho(a,Ta) + \rho(Ta,T^2a)]$$
$$\le (\alpha+\beta+\delta)\rho(a,Ta) + (\gamma+\delta)\rho(Ta,T^2a)$$

or

$$\rho(Ta,T^2a) \le \left(\frac{\alpha+\beta+\delta}{1-(\gamma+\delta)}\right)\rho(a,Ta). \quad (2.2)$$

Since $\frac{\alpha+\beta+\delta}{1-(\gamma+\delta)} < 1$, from (2.2), it would follow that $\rho(Ta,T^2a) < l$ where $\rho(Ta,T^2a) \in D$. In other words, l cannot be a lower bound of D, which is a contradiction.

Therefore, $l = \inf \mathrm{D} = 0$.

By Lemma 2.2 , we can choose the points $a_1, a_2, \ldots, a_n, \ldots$ in Y such that

$$\rho(a_n, Ta_n) \in D \text{ for } n = 1,2,3,\ldots \text{ and } \lim_{n\to\infty} \rho(a_n, Ta_n) = 0. \quad (2.3)$$

We now prove that $\langle a_n \rangle_{n=1}^{\infty}$ is Cauchy.

Using (2.1) with $a = a_n, b = a_m$ and the trianangle inequality (d3), we get

$$\rho(a_n,a_m) \le \rho(a_n,Ta_n) + \rho(Ta_n,a_m)$$
$$\le \rho(a_n,Ta_n) + \rho(Ta_n,Ta_m) + \rho(Ta_m,a_m)$$
$$= \rho(a_n,Ta_n) + \rho(a_m,Ta_m) + \rho(Ta_n,Ta_m)$$
$$\le \rho(a_n,Ta_n) + \rho(a_m,Ta_m) + \alpha\rho(a_n,a_m) + \beta\rho(a_n,Ta_n) + \gamma\rho(a_m,Ta_m)$$
$$+\delta \max\left\{\rho(a_n,Ta_m), \rho(a_m,Ta_n), \rho(a_n,Ta_n), \rho(a_m,Ta_m)\right\}$$
$$\le \rho(a_n,Ta_n) + \rho(a_m,Ta_m) + \alpha\rho(a_n,a_m) + \beta\rho(a_n,Ta_n) + \gamma\rho(a_m,Ta_m)$$
$$+\delta \max\left\{\rho(a_n,a_m) + \rho(a_m,Ta_m), \rho(a_m,a_n) + \rho(a_n,Ta_n), \rho(a_n,Ta_n), \rho(a_m,Ta_m)\right\}$$

Employing the limit as $n \to \infty$ in this and using (2.3),

$$\lim_{n\to\infty}\rho(a_n,a_m)=0 \quad \text{for all} \quad m\geq n. \tag{2.4}$$

proving that $\langle a_n\rangle_{n=1}^{\infty}$ is Cauchy.

Since Y is complete, we can find a point $z\in Y$ such that

$$\lim_{n\to\infty} y_n = z. \tag{2.5}$$

Again, by triangle inequality (d3) and using (2.1) with $a=a_n, b=z$, we have

$$\begin{aligned}\rho(z,Tz)&\leq \rho(z,Ta_n)+\rho(Ta_n,Tz)\\ &\leq[\rho(z,a_n)+\rho(a_n,Ta_n)]+\alpha\rho(a_n,z)+\beta\rho(a_n,Ta_n)+\gamma\rho(z,Tz)\\ &\quad+\delta\max\{\rho(a_n,Tz),\rho(z,Ta_n),\rho(a_n,Ta_n),\rho(z,Tz)\}\\ &\leq[\rho(z,a_n)+\rho(a_n,fa_n)]+\alpha\rho(a_n,z)+\beta\rho(a_n,fa_n)+\gamma\rho(z,fz)\\ &\quad+\delta\max\{\rho(a_n,Tz),\rho(z,Ta_n),\rho(a_n,Ta_n),\rho(z,Tz)\}\\ &\leq[\rho(z,a_n)+\rho(a_n,Ta_n)]+\alpha\rho(a_n,z)+\beta\rho(a_n,Ta_n)+\gamma\rho(z,Tz)\\ &\quad+\delta\max\{\rho(a_n,z)+\rho(z,Tz),\rho(z,a_n)+\rho(a_n,Ta_n),\rho(a_n,Ta_n),\rho(z,Tz)\}\end{aligned}$$

Now applying the limit as $n\to\infty$ in this and using (2.3) and (2.5), we have

$$\rho(z,Tz)\leq(\gamma+\delta)\rho(z,Tz)$$

or

$(1-\gamma-\delta)d(z,Tz)\leq 0$ so that $d(z,Tz)=0$, which yields $Tz=z$.

That is, z is a fixed point of T.

Uniqueness: Suppose t is another fixed point of T. That is $Tt=t$.

Then from (2.1) with $a=z$ and $b=t$, we have

$$\begin{aligned}\rho(z,t)&=\rho(Tz,Tt)\\ &\leq\alpha\rho(z,t)+\beta\rho(z,Tz)+\gamma\rho(t,Tt)\\ &\quad+\delta\max\{\rho(z,Tt),\rho(t,fz),\rho(z,Tz),\rho(t,Tt)\}\\ &\leq\alpha\rho(z,t)+\beta\rho(z,z)+\gamma\rho(t,t)\\ &\quad+\delta\max\{\rho(z,t),\rho(t,z),\rho(z,z),\rho(t,t)\}\end{aligned}$$

or $(1-\alpha-\delta)\rho(z,t)\leq 0$ so that $\rho(z,t)=0$ which implies that $z=t$.

That is, z is a unique fixed point of T.

This completes the proof.

Remark 2.1: It is significant to establish that z is a contractive fixed point of T.

Indeed taking $a=T^{n-1}a$ and $b=z$ in (2.1) and using triangle inequality (d3), we get

$$\begin{aligned}\rho(T^na,z)&=d(T^na,Tz)\\ &\leq\alpha\rho(T^{n-1}a,z)+\beta\rho(T^{n-1}a,TT^{n-1}a)+\gamma\rho(zp,Tz)\\ &\quad+\delta\max\{\rho(T^{n-1}a,Tz),\rho(z,TT^{n-1}a),\rho(T^{n-1}a,TT^{n-1}a),\rho(z,Tz)\}\\ &=\alpha\rho(T^{n-1}a,z)+\beta\rho(T^{n-1}a,T^na)+\gamma\rho(z,z)\\ &\quad+\delta\max\{\rho(T^{n-1}a,z),\rho(z,T^na),\rho(T^{n-1}a,T^na),\rho(z,z)\}\\ &\leq\alpha[\rho(T^{n-1}a,T^na)+\rho(T^na,z)]+\beta\rho(fT^{n-1}a,T^na)+\gamma\rho(z,z)\\ &\quad+\delta\max\{[\rho(T^{n-1}a,T^na)+\rho(T^na,z)],\rho(z,T^na),\rho(T^{n-1}a,T^na),\rho(z,z)\}\end{aligned}$$

$$=\alpha[\rho(T^{n-1}a,T^n a)+\rho(T^n a,z)]+\beta\rho(T^{n-1}a,T^n a)+\delta[\rho(T^{n-1}a,T^n a)+\rho(T^n a,z)]$$

or

$$\rho(T^n a,z)\le\frac{\alpha+\beta+\delta}{1-\alpha-\delta}\cdot\rho(T^{n-1}a,T^n a) \qquad (2.6)$$

Again by taking $a=T^{n-2}a$, $b=T^{n-1}a$ in (2.1) and using triangle inequality(d3),we have

$$\begin{aligned}\rho(T^{n-1}a,T^n a)&=\rho(TT^{n-2}a,TT^{n-1}a)\\&\le\alpha\rho(T^{n-2}a,T^{n-1}a)+\beta\rho(T^{n-2}a,TT^{n-2}a)+\gamma\rho(T^{n-1}a,TT^{n-1}a)\\&\quad+\delta\max\left\{\rho(T^{n-2}a,TT^{n-1}a),\rho(T^{n-1}a,TT^{n-2}a),\rho(T^{n-2}a,TT^{n-2}a),\rho(T^{n-1}a,TT^{n-1}a)\right\}\\&=\alpha\rho(T^{n-2}a,T^{n-1}a)+\beta\rho(T^{n-2}a,T^{n-1}a)+\gamma\rho(T^{n-1}a,T^n a)\\&\quad+\delta\max\left\{\rho(T^{n-2}a,T^n a),\rho(T^{n-1}a,T^{n-1}a),\rho(T^{n-2}a,T^{n-1}a),\rho(T^{n-1}a,T^n a)\right\}\\&\le\alpha\rho(T^{n-2}a,T^{n-1}a)+\beta\rho(T^{n-2}a,T^{n-1}a)+\gamma\rho(T^{n-1}a,T^n a)\\&\quad+\delta\max\left\{[\rho(T^{n-2}a,T^{n-1}a)+\rho(T^{n-1}a,T^n a)],\rho(T^{n-2}a,T^{n-1}a),\rho(T^{n-1}a,T^n a)\right\}\\&=\alpha\rho(T^{n-2}a,T^{n-1}a)+\beta\rho(T^{n-2}a,T^{n-1}a)+\gamma\rho(T^{n-1}a,T^n a)\\&\quad+\delta[\rho(T^{n-2}a,T^{n-1}a)+\rho(T^{n-1}a,T^n a)]\end{aligned}$$

or

$$\rho(T^{n-1}a,T^n a)\le\frac{\alpha+\beta+\delta}{1-\gamma-\delta}\cdot\rho(T^{n-2}a,T^{n-1}a).$$

By induction, we have

$$\rho(T^{n-1}a,T^n a)\le\left[\frac{\alpha+\beta+\delta}{1-\gamma-\delta}\right]^{n-1}\cdot\rho(a,Ta).$$

Substituting this in (2.6), we get

$$\rho(T^n a,z)\le\frac{\alpha+\beta+\delta}{1-\alpha-\delta}\left[\frac{\alpha+\beta+\delta}{1-\gamma-\delta}\right]^{n-1}\cdot\rho(a,Ta).$$

Since $\dfrac{\alpha+\beta+\delta}{1-\gamma+\delta}<1$, which as $n\to\infty$ gives $T^n a\to z$ for each $y\in Y$.

Thus z is a contractive fixed point of T.

Theorem 2.2. *Let T be a self-map on complete metric space (Y,ρ) such that*

$$\rho(Ta,Tb)\le h\cdot\max\left\{T(a,b),\frac{\rho(a,Ta)+\rho(b,Tb)}{2},\frac{\rho(a,Tb)+\rho(b,Tb)}{2}\right\}\text{ for all } a,b\in Y, \quad (2.7)$$

where $0\le h<1$. Then there exists a unique fixed point of T in Y.

Proof. Let $D=\{\rho(a,Ta):a\in Y\}$.

It is clear that D is a nonempty set of nonnegative real numbers which is bounded below. Hence by Lemma 2.1, it has the infimum, say $l\ge 0$.

We claim that $l=0$.

If possible, we suppose that $l>0$. Now from (2.7) with $b=Ta$ and the trianangle inequality (d3), we have

$$\rho(Ta,T^2a)\le h\cdot\max\left\{\rho(a,Ta),\frac{\rho(a,Ta)+\rho(Ta,T^2a)}{2},\frac{\rho(a,T^2a)+\rho(Ta,Ta)}{2}\right\}$$

$$\leq h\cdot\max\left\{\rho(a,Ta),\frac{\rho(a,Ta)+\rho(Ta,T^2a)}{2},\frac{\rho(a,T^2a)+\rho(Ta,Ta)}{2}\right\}$$

$$=h\cdot\max\left\{\rho(a,Ta),\frac{\rho(a,Ta)+\rho(Ta,T^2a)}{2}\right\}$$

$$\rho(Ta,T^2a)\leq hM \quad \text{where} \quad M=\max\left\{\rho(a,Ta),\frac{\rho(a,Ta)+\rho(Ta,T^2a)}{2}\right\} \qquad (2.8)$$

Case 1:Suppose that $M=\rho(a,Ta)$, then we have

$$\rho(Ta,T^2a)\leq h\rho(a,Ta)<\rho(a,Ta)$$

Case 2: Suppose that $M=\dfrac{\rho(a,Ta)+\rho(Ta,T^2a)}{2}$, then

$$\rho(Ta,T^2a)\leq h\frac{\rho(a,Ta)+\rho(Ta,T^2a)}{2}$$

$$=\frac{h}{2-h}\rho(a,Ta)$$

$$<\rho(a,Ta)$$

From Case 1 and Case 2, we observe that $\rho(Ta,T^2a)<\rho(a,Ta)$.

Hence it follows that $\rho(Ta,T^2a)<a$ where $\rho(Ta,T^2a)\in D.$ which is a contradiction.

Hence $l=\inf D=0$.

By Lemma 2.2, there exist $a_1,a_2,\ldots,a_n,\ldots$ in Y such that

$$\rho(a_n,Ta_n)\in S \text{ for } n=1,2,3,\ldots \text{ and } \lim_{n\to\infty}\rho(a_n,Ta_n)=0. \qquad (2.9)$$

We now prove that $\langle a_n\rangle_{n=1}^{\infty}$ is Cauchy.

Utilizing (2.7) with $a=a_n, b=a_m$ and the trianangle inequality (d3), we get

$$\rho(a_n,a_m)\leq\rho(a_n,Ta_n)+\rho(Ta_n,a_m)$$

$$\leq\rho(a_n,Ta_n)+\rho(Ta_n,Ta_m)+\rho(Ta_m,a_m)$$

$$=\rho(a_n,Ta_n)+\rho(a_m,Ta_m)+\rho(Ta_n,Ta_m)$$

$$\leq\rho(a_n,Ta_n)+\rho(a_m,Ta_m)+h\cdot\max\left\{\rho(a_n,a_m),\frac{\rho(a_n,Ta_n)+\rho(a_m,Ta_m)}{2}\right.$$
$$\left.\frac{\rho(a_n,Ta_m)+\rho(a_m,Ta_n)}{2}\right\}$$

$$\leq\rho(a_n,Ta_n)+\rho(a_m,Ta_m)+h\cdot\max\left\{\rho(a_n,a_m),\frac{\rho(a_n,Ta_n)+\rho(a_m,Ta_m)}{2},\right.$$
$$\left.\frac{\rho(a_n,a_m)+\rho(a_m,Ta_m)+\rho(a_m,a_n)+\rho(a_n,Ta_n)}{2}\right\}$$

Letting $n\to\infty$ in this and using (2.9),

$$\lim_{n\to\infty}d(a_n,a_m)=0 \quad \text{for all} \quad m\geq n. \qquad (2.10)$$

proving that $\langle a_n\rangle_{n=1}^{\infty}$ is Cauchy. Since Y is complete, we can find a point $z\in Y$ such that

$$\lim_{n\to\infty} a_n = z. \qquad (2.11)$$

Now by triangle inequality (d3) and using (2.7) with $a = a_n, b = z$, we have

$$\rho(z,Tz) \le \rho(z,Ta_n) + \rho(Ta_n,Tz)$$

$$\le [\rho(p,a_n) + \rho(a_n,Ta_n)] + h\cdot\max\left\{\rho(a_n,z), \frac{\rho(a_n,Ta_n)+\rho(z,Tz)}{2}, \frac{\rho(a_n,Tz)+\rho(z,Ta_n)}{2}\right\}$$

$$\le [\rho(z,a_n) + \rho(a_n,Ta_n)]$$

$$+h\cdot\max\left\{\rho(a_n,z), \frac{\rho(a_n,Ta_n)+\rho(z,Tz)}{2}, \frac{\rho(a_n,z)+\rho(z,Tz)+\rho(z,a_n)+\rho(a_n,Ta_n)}{2}\right\}$$

Now employing the limit as $n \to \infty$ in this and using (2.9) and (2.11), we have

$$\rho(z,Tz) \le h\rho(z,Tz)$$

Or $(1-h)\rho(z,Tz) \le 0$ so that $\rho(z,Tz) = 0$, which implies that $Tz = z$.

That is, z is a fixed point of T.

Uniqueness: Suppose t is a point in Y such that $Tt = t$. Then from (2.7) with $a = z$ and $b = t$, we have

$$\rho(z,t) = \rho(Tz,Tt)$$

$$\le h\cdot\max\left\{\rho(z,t), \frac{\rho(z,Tz)+\rho(t,Tt)}{2}, \frac{\rho(z,Tt)+\rho(t,Tz)}{2}\right\}$$

$$\le h\cdot\max\left\{\rho(z,t), \frac{\rho(z,Tz)+\rho(t,Tt)}{2}, \frac{\rho(z,t)+\rho(t,Tt)+\rho(t,z)+\rho(z,Tz)}{2}\right\}$$

or $(1-h)\rho(z,t) \le 0$ so that $\rho(z,t) = 0$ which yields that $z = t$.

That is, T has a unique fixed point z in Y.

Remark 2.2. It is significant to establish that z is a contractive fixed point of T.

Now taking $a = T^{n-1}a$ and $b = z$ in (2.7) and using triangle inequality(d3), we get

$$\rho(T^n a,z) = \rho(T^n a,Tz)$$

$$\le h\cdot\max\left\{\rho(T^{n-1}a,z), \frac{\rho(T^{n-1}a,TT^{n-1}a)+\rho(z,Tz)}{2}, \frac{\rho(T^{n-1}a,Tz)+\rho(z,TT^{n-1}a)}{2}\right\}$$

$$= h\cdot\max\left\{\rho(T^{n-1}a,z), \frac{\rho(T^{n-1}a,T^n a)+\rho(z,z)}{2}, \frac{\rho(T^{n-1}a,Tz)+\rho(z,T^n a)}{2}\right\}$$

$$\le h\cdot\max\left\{\rho(T^{n-1}a,z), \frac{\rho(T^{n-1}a,z)+\rho(z,T^n a)}{2}, \frac{\rho(T^{n-1}a,z)+\rho(z,T^n a)}{2}\right\}$$

$$= h\cdot\max\left\{\rho(T^{n-1}a,z), \frac{\rho(T^{n-1}a,z)+\rho(z,T^n a)}{2}\right\} = kM_1$$

Suppose that $M_1 = \rho(T^{n-1}a,z)$, then $\rho(T^n a,z) \le c_1\cdot\rho(T^{n-1}a,z)$ where $c_1 = h < 1$.

Suppose that $M_1 = \dfrac{\rho(T^{n-1}a,z)+\rho(z,T^n a)}{2}$, then $\rho(T^n a,z) \le c_2\cdot\rho(T^{n-1}a,z)$, where $c_2 = \dfrac{k}{2-k} < 1$.

Let $c = max\{c_1,c_2\}$. Then we have $\rho(T^n a,z) \le c\cdot\rho(T^{n-1}a,z)$, where $c < 1$.

By induction, we have $\rho(T^n a,z) \le c^n\rho(Ta,z)$, which as $n \to \infty$ gives $T^n a \to z$ for each $z \in Y$.

That is, z is a contractive fixed point of T.

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