Generalized Fixed Point Theorems on Complex Partial b-Metric Space

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ABSTRACT: : *In this paper , we obtain a unique fixed point theorem on complex partial b-metric space which is generalized results of [1]. Also an example is to demonstrate our result.*

Key Words: *complex valued b-metric space; complex partial b-metric space; fixed point.*

1. Introduction and Preliminaries

In many branches of science, economics, computer science, engineering and thedevelopment of nonlinear analysis, the fixed point theory is one of the most important tool. In 1989, Backhtin [2]introduced the concept of b-metric space. In 1993, Czerwik [3] extended the results of b-metric spaces. Azam et al. [4] introduced new spaces called complex valued metric spaces and established the existence of fixed point theorems under the contraction condition. Rao et al. [5] introduced the concept of complex valued b-metric space which was more general than the well known complex valued metric space. P. Dhivya and M. Marudai [6] introduced new spaces called complex partial metric space and established the existence of common fixed point theorems under the contraction condition of rational expression. In this paper, we obtain a unique fixed point theorem on complex partial b-metric space which is generalized results of [1].

We recall some basic notions and definitions which will be useful for proving our main results.
Let C be the set of Complex numbers and $z_1, z_2 \in C$. Define partial order ≤ on C as follows.
$z_1 \leq z_2$ if and only if Re.(z_1)≤ Re.(z_2) also Im.(z_1)≤ Im.(z_2)
It follows that $z_1 \leq z_2$ if one of the following condition hold

1. $Re.(z_1) = Re.(z_2)$, $Im.(z_1) < Im.(z_2)$
2. $Re.(z_1) < Re.(z_2)$, $Im.(z_1) = Im.(z_2)$
3. $Re.(z_1) < Re.(z_2)$,$Im.(z_1) < Im.(z_2)$
4. $Re.(z_1) = Re.(z_2)$, $Im.(z_1) = Im.(z_2)$

In particular, we will write $z_1 \prec z_2$ if $z_1 \neq z_2$ and one of (1), (2) and (4) is satisfied and we will write $z_1 < z_2$ if only (3) is satisfied.

Definition [1]

A complex partial b-metric on a non-empty set X is a function $p_{cb} : X \times X \to C^+$ such that for all $x, y, z \in X$:

(i) $0 \leq p_{cb}(x,x) \leq p_{cb}(x,y)$ (small self-distances)

(ii) $p_{cb}(x,y) = p_{cb}(y,x)$ (symmetry)

(iii) $p_{cb}(x,x) = p_{cb}(x,y) = p_{cb}(y,y)$ if and only if x = y(equality)

(iv) $\exists$ a real number $s \geq 1$ such that $p_{cb}(x,y) \leq s\left(p_{cb}(x,z) + p_{cb}(z,y)\right) - p_{cb}(z,z)$ (triangularity).

A complex partial b-metric is a pair (X, p_{cb}) such that X is a nonempty set and p_{cb} is a complex partial b-metric on X . The number s is called the coefficient of (X, p_{cb}).

Remark [1].

In a complex partial b-metric space (X, p_{cb}) if $x, y \in X$ and $p_{cb}(x, y) = 0$, then $x = y$, but the converse may not be true.

Remark[1].

It is clear that every complex partial metric space is a complexpartial b-metric space with coefficient $s=1$ and every complex valued b-metric isa complex partial b-metric space with the same coefficient and zero self-distance.

However, the converse of this fact need not hold.

Definition [1]

Let (X, p_{cb}) be a complex partial b-metric space with coefficient s. Let $\{x_n\}$ be any sequence in X and $x \in X$. Then:

(i) The sequence $\{x_n\}$ is said to be convergent with respect to τ_{cb} and convergesto x, if $\lim_{n\to\infty} p_{cb}(x_n, x) = p_{cb}(x, x)$.

(ii) The sequence $\{x_n\}$ is said to be Cauchy sequence in (X, p_{cb}) if $\lim_{n\to\infty} p_{cb}(x_n, x_m)$ exists and is finite.

(iii) (X, p_{cb}) is said to be a complete complex partial b-metric space if for every Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that $\lim_{n,m\to\infty} p_{cb}(x_n, x_m) = \lim_{n\to\infty} p_{cb}(x_n, x) = p_{cb}(x, x)$.

(iv) A mappings $R: X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $R(B_{p_{cb}}(x_0, \delta)) \subset B_{p_{cb}}(R(x_0, \varepsilon))$.

Example[1]

Let $Y = R^+$, $q > 1$ a constant and $p_{cb}: Y \times Y \to C^+$ be defined by

$p_{cb}(y,l) = ([\max\{y,l\}]^q + |y-l|^q)(1+i)$ for all $y, l \in Y$.

Then (Y, p_{cb}) is a complete complex partial b-metric space with coefficient $s = 2^q > 1$, but it is neither a b-metric nor a partial metric space.

2.Main Results

Theorem1

Let (R, p_{cb}) be a complete complex partial b-metric space with coefficient $s \geq 1$ and $Q: R \to R$ be a mapping satisfying the following condition:

$p_{cb}(Qr, Qp) \leq \alpha[p_{cb}(r, Qr) + p_{cb}(p, Qp)]$ for all $r, p \in R$,

where $\alpha \in [0, \frac{1}{s})$. Then Q has a unique fixed point $q \in R$ and $p_{cb}(q, q) = 0$.

Proof.

Let $r_0 \in R$ be arbitrary, define sequence $\{r_n\}$ in R such that $r_n = Rr_{n-1}$.

For any $n \in N$,

$$p_{cb}(r_n, r_{n+1}) = p_{cb}(Qr_{n-1}, Qr_n)$$

$$\leq \alpha[p_{cb}(r_{n-1}, Qr_{n-1}) + p_{cb}(r_n, Qr_n)]$$

$$= \alpha[p_{cb}(r_{n-1}, r_n) + p_{cb}(r_n, r_{n+1})]$$

$$p_{cb}(r_n, r_{n+1}) \leq \left(\frac{\alpha}{1-\alpha}\right)(p_{cb}(r_{n-1}, r_n)), \text{ where } h = \frac{\alpha}{1-\alpha} < 1$$

Then it follows that

$$p_{cb}(r_n, r_{n+1}) \leq h p_{cb}(r_{n-1}, r_n) \leq \dots\dots\dots\dots \leq h^n p_{cb}(r_0, r_1).$$

For any $n, m \in N$,with $m > n$,

$p_{cb}(r_n,r_m) \le s[p_{cb}(r_n,r_{n+1}) + p_{cb}(r_{n+1},r_m)] - p_{cb}(r_{n+1},r_{n+1})$

$\le sp_{cb}(r_n,r_{n+1}) + s^2[p_{cb}(r_{n+1},r_{n+2}) + p_{cb}(r_{n+2},r_m)] - sp_{cb}(r_{n+2},r_{n+2})$

$\le sp_{cb}(r_n,r_{n+1}) + s^2 p_{cb}(r_{n+1},r_{n+2}) + s^3 p_{cb}(r_{n+2},r_{n+2}) + ... + s^{m-n} p_{cb}(r_{m-1},r_m)$

$\le sh^n p_{cb}(r_1,r_0) + s^2h^{n+1} p_{cb}(r_1,r_0) + s^3h^{n+2} p_{cb}(r_1,r_0) + ... + s^{m-n}h^{m-1} p_{cb}(r_1,r_0)$

$= sh^n[1 + sh + (sh)^2 + ...] p_{cb}(r_1,r_0)$

$= \dfrac{sh^n}{1-sh}\left(p_{cb}(r_1,r_0)\right).$

Thus,

$\left|p_{cb}(r_m,r_n)\right| \le \dfrac{sh^n}{1-sh}\left|p_{cb}(r_1,r_0)\right| \to 0$

as $m,n \to \infty$ which implies that $\lim_{n,m\to\infty} p_{cb}(r_n,r_m) = 0$ such that $\{r_n\}$ is a Cauchy sequence in R.

By completeness of R there exists $q \in R$ such that

$\lim_{n\to\infty} p_{cb}(r_n,q) = \lim_{n,m\to\infty} p_{cb}(r_n,r_m) = p_{cb}(q,q) = 0$ ....(1)

Next, we have to prove q is a fixed point of Q.

Suppose that q is not a fixed point of Q, then $p_{cb}(q,Qq) > 0$.

For any $n \in N$,

$p_{cb}(q,Qq) \le s[p_{cb}(q,r_{n+1}) + p_{cb}(r_{n+1},Qq)] - p_{cb}(r_{n+1},r_{n+1})$

$\le s[p_{cb}(q,r_{n+1}) + p_{cb}(Qr_n,Qq)]$

$\le s[p_{cb}(q,r_{n+1}) + \alpha\{p_{cb}(r_n,Qr_n) + p_{cb}(q,Qq)\}]$

$\le \dfrac{s}{1-s\alpha} p_{cb}(q,r_{n+1}) + \dfrac{s\alpha}{1-s\alpha} p_{cb}(r_n,r_{n+1})$

$|p_{cb}(q,Qq)| \le \dfrac{s}{1-s\alpha}|p_{cb}(q,r_{n+1})| + \dfrac{s\alpha}{1-s\alpha}|p_{cb}(r_n,r_{n+1})|$

As $n \to \infty$,

$|p_{cb}(q,Qq)| \le 0$, which is a contradiction.

Therefore q is a fixed point of Q.

To prove the uniqueness of fixed point .

Suppose $a,b \in R$ be two distinct points of Q.

Then $p_{cb}(a,a) = p_{cb}(b,b) = 0$.

Now,

$p_{cb}(a,b) = p_{cb}(Qa,Qb) \le \alpha[p_{cb}(a,Qa) + p_{cb}(b,Qb)]$

$= \alpha[p_{cb}(a,a) + p_{cb}(b,b)] = 0$.

$|p_{cb}(a,b)| \le 0$, which is a contradiction.

Therefore q is a unique fixed point of Q

This complete the proof.

Theorem 2

Let (R, p_{cb}) be any complete complex partial b-metric space with coefficient $s \ge 0$ and $Q: R \to R$ be a mapping satisfying:

$p_{cb}(Rr, Rp) \le \alpha \max\{p_{cb}(r,p), p_{cb}(r,Rr), p_{cb}(p,Rp)\}$ for all $r, p \in R$, where $\alpha \in [0, \frac{1}{s})$. Then Q has a unique fixed point $q \in R$ and $p_{cb}(q,q) = 0$.

Proof

Let $r_0 \in R$ be arbitrary, define sequence $\{r_n\}$ in R such that $r_n = Rr_{n-1}$.

For any $n \in N$,

$$p_{cb}(r_{n+1}, r_n) = p_{cb}(Rr_n, Rr_{n-1})$$
$$\leq \alpha \max\{p_{cb}(r_n, r_{n-1}), p_{cb}(r_n, Rr_n), p_{cb}(r_{n-1}, Rr_{n-1})\}$$
$$= \alpha \max\{p_{cb}(r_n, r_{n-1}), p_{cb}(r_n, r_{n+1}), p_{cb}(r_{n-1}, r_n)\}$$
$$= \alpha \max\{p_{cb}(r_n, r_{n-1}), p_{cb}(r_n, r_{n+1})\}$$

Suppose that $\max\{p_{cb}(r_n, r_{n-1}), p_{cb}(r_n, r_{n+1})\} = p_{cb}(r_n, r_{n+1})$.

Then

$p_{cb}(r_{n+1}, r_n) < p_{cb}(r_{n+1}, r_n)$, which is a contradiction.

Therefore, $\max\{p_{cb}(r_n, r_{n-1}), p_{cb}(r_n, r_{n+1})\} = p_{cb}(r_n, r_{n-1})$.

Then $p_{cb}(r_n, r_{n+1}) \leq \alpha p_{cb}(r_n, r_{n-1})$.

Continuing this process, we obtain

$p_{cb}(r_n, r_{n+1}) \leq \alpha^n p_{cb}(r_1, r_0)$ for all $n \geq 0$.

For any $n, m \in N$,with $m > n$,

$$p_{cb}(r_n, r_m) \leq s[p_{cb}(r_n, r_{n+1}) + p_{cb}(r_{n+1}, r_m)] - p_{cb}(r_{n+1}, r_{n+1})$$
$$\leq sp_{cb}(r_n, r_{n+1}) + s^2[p_{cb}(r_{n+1}, r_{n+2}) + p_{cb}(r_{n+2}, r_m)] - sp_{cb}(r_{n+2}, r_{n+2})$$
$$\leq sp_{cb}(r_n, r_{n+1}) + s^2 p_{cb}(r_{n+1}, r_{n+2}) + s^3 p_{cb}(r_{n+2}, r_{n+2}) + ... + s^{m-n} p_{cb}(r_{m-1}, r_m)$$
$$\leq sh^n p_{cb}(r_1, r_0) + s^2 h^{n+1} p_{cb}(r_1, r_0) + s^3 h^{n+2} p_{cb}(r_1, r_0) + ... + s^{m-n} h^{m-1} p_{cb}(r_1, r_0)$$
$$= sh^n[1 + sh + (sh)^2 + ...] p_{cb}(r_1, r_0)$$
$$= \frac{sh^n}{1 - sh}\left(p_{cb}(r_1, r_0)\right).$$

Thus,

$$\left|p_{cb}(r_m, r_n)\right| \leq \frac{sh^n}{1 - sh}\left|p_{cb}(r_1, r_0)\right| \to 0$$

as $m, n \to \infty$ which implies that $\lim_{n,m\to\infty} p_{cb}(r_n, r_m) = 0$ such that $\{r_n\}$ is a Cauchy sequence in R.

By completeness of R there exists $q \in R$ such that

$$\lim_{n\to\infty} p_{cb}(r_n, q) = \lim_{n,m\to\infty} p_{cb}(r_n, r_m) = p_{cb}(q, q) = 0 \text{ ....(2)}$$

Next, we have to prove q is a fixed point of Q.

Suppose that q is not a fixed point of Q, then $p_{cb}(q, Qq) > 0$.

For any $n \in N$,

$$p_{cb}(q, Qq) \leq s[p_{cb}(q, r_{n+1}) + p_{cb}(r_{n+1}, Qq)] - p_{cb}(r_{n+1}, r_{n+1})$$
$$\leq s[p_{cb}(q, r_{n+1}) + p_{cb}(Qr_n, Qq)]$$
$$\leq sp_{cb}(q, r_{n+1}) + s\alpha p_{cb}(r_n, q)$$
$$| p_{cb}(q, Qq) | \leq s | p_{cb}(q, r_{n+1}) | + s\alpha | p_{cb}(r_n, r_{n+1}) |$$

As $n \to \infty$,

$| p_{cb}(q, Qq) | \leq 0$, which is a contradiction.

Therefore q is a fixed point of Q.
To prove the uniqueness of fixed point.
Suppose $a,b \in R$ be two distinct points of Q.
Then $p_{cb}(a,a) = p_{cb}(b,b) = 0$.
Now,

$$p_{cb}(a,b) = p_{cb}(Qa,Qb) \le \alpha \max\{p_{cb}(a,b), p_{cb}(a,Qa), p_{cb}(b,Qb)\}$$

$$= \alpha \max\{p_{cb}(a,b), p_{cb}(a,a), p_{cb}(b,b)\}.$$

$$= \alpha\, p_{cb}(a,b)$$

$$< p_{cb}(a,b)$$

$| p_{cb}(a,b) | < | p_{cb}(a,b) |$, which is a contradiction.
Therefore q is a unique fixed point of Q
This complete the proof.

Example
Let $Y = R^{+}$, $q > 1$ a constant and $p_{cb} : Y \times Y \to C^{+}$ be defined by

$p_{cb}(y,l) = ([\max\{y,l\}]^{q} + | y - l |^{q})(1+i)$ for all $y,l \in Y$.

Then (Y, p_{cb}) is a complete complex partial b-metric space with coefficient $s = 2^{q} > 1$, but it is neither a b-metric nor a partial metric space.

Define $Q : Y \to Y$ by $Qy = \dfrac{y}{3}$, for all $y \in Y$

Let $q = 2$.
Now,

$$p_{cb}\left(Q1,Q2\right) = p_{cb}\left(\frac{1}{3},\frac{2}{3}\right) = \left(\frac{4}{9}+\frac{1}{9}\right)\left(1+i\right) = \left(\frac{5}{9}\right)\left(1+i\right)$$

$$\left(\frac{1}{2}\right)\left(p_{cb}(1,Q1) + p_{cb}(2,Q2)\right) = \left(\frac{1}{2}\right)\left(1+\frac{4}{9}+4+\frac{16}{9}\right)\left(1+i\right) = \left(\frac{65}{9}\right)\left(1+i\right)$$

Thus $p_{cb}\left(Q1,Q2\right) \le \alpha[p_{cb}(1,Q1) + p_{cb}(2,Q2)]$.

Therefore all the condition of Theorem 1 is satisfied. Clearly 0 is the unique fixed point of Q.

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