A Review of a class of Nonuniform Wavelets

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ABSTRACT: *Wavelet packets are a class of generalized wavelets having applications in many fields including signal processing, image compression and solving integral equations. In this present paper we have considered a generalization of the notion of nonuniform multiresolution analysis (NUMRA) which is called vector-valued nonuniform multiresolution analysis (VNUMRA). We have reviewed vector-valued nonuniform multiresolution analysis(VNUMRA) where the associated subspace V_0 of $L^2(R;C^s)$ has, an orthonormal basis, a collection of translates of a vector valued function over the set $\Lambda = \{0, r/N\} + 2Z$, where r and N are relatively prime and $1 \le r \le 2N-1$ and the corresponding dilation factor is 2N. The notion of vector-valued nonuniform wavelet packets is introduced in this paper and their various properties are investigated. The orthonormal basis of $L^2(R;C^s)$ is constructed from these wavelet packets and the orthonormal decomposition relation is also given.*
Keywords: *Nonuniform multiresolution analysis, vector-valued wavelets, wavelets, wavelet packets..*

Introduction

Wavelet analysis is a refinement of Fourier analysis and has been introduced by practical considerations in 1980's. The big disadvantage of Fourier expansion was that it had only frequency resolution and no time resolution. Wavelet analysis has provided a new class of orthogonal expansions in $L^2(R)$ with good time-frequency and regularity approximation properties which have been successfully applied to many areas like signal processing, numerical analysis, approximation theory, computer science and engineering. Wavelet analysis is an important rudiments of applied and numerical harmonic analysis. Wavelet system is generated by the action of translations and dilations on a single or finite family of functions in L^2 (R). Both of these theories are outcome of the integrative efforts that brought mathematicians, physicists and engineers together. This connection has generated a flow of ideas that exceeds the erection of new bases or transforms.

Reproducing system of functions including Wavelets have been applied successfully in a variety of applications in different areas of science and technology. A general reproducing system known as composite wavelets which are defined with composite dilations provide truly multidimensional generalizations of traditional wavelets. . In dimension two, the elements of such systems are defined not only at various scales and locations as traditional wavelet system but also at various orientations or angles. A special case of composite wavelet system is called shearlet system.

The domain of this field is growing up very fast. There are several research papers and books say for example by Demetrio Labate, Guido Weiss, Ingrid Daubechies and A.H. Siddiqi and references there in. Extensive study of reproducing system has been carried out by Shearlet group which includes Gitta Kutynoik, De Labate, Tomes Sauer etc.

In the recent years, Wavelet packets and Wave packet system have been introduced. Wavelet packets are extensions of wavelets which yield basis functions with better frequency localization. Wave packet is again a reproducing system generated by the combined action of a class of translations, modulations and dilations on a finite family of functions. Wave packet contains Wavelet system and Gabor system as special cases. Ole Christensen and Asghar Rahimi have studied frame properties of wave packet systems in $L^2(R^d)$.

The concept of vector valued multiresolution analysis (MRA) and associated wavelets have been studied. The necessary and sufficient conditions for existence of compactly supported orthogonal vector valued wavelets have been obtained. Vector valued wavelet packets have been introduced recently and some properties have been investigated. Farkov has studied wavelets and associated multiresolution analysis on half line. The Haar wavelet based on Haar system, introduced by the Hungarian mathematician Alfred Haar in 1909[1], is the simplest example of wavelets.

2. Vector- Valued Wavelets and Wavelet Packets

WAVELET packets are a class of generalized wavelets having applications in many fields including signal processing, image compression and solving integral equations. For wavelet basis, all the high- frequency wavelets have poor frequency localization. Wavelet packets were introduced by R.Coifman, Y.Meyer and M.V.Wickerhauser [2], [3] in order to improve the frequency resolution and thereby get more efficient algorithms to decompose signals containing both transient and stationary components.

The multiresolution analysis is known as the heart of wavelet theory. The concept of multiresolution provides a very elegant tool for the construction of wavelets i.e. the function $\psi \in L^2(R)$ having the property that the collection of functions $\{2^{j/2}\psi(2^j x-n)\}_{j,n\in Z}$ forms a complete orthonormal system for $L^2(R)$, where Z is the set of integers. A multiresolution analysis on the real line R, introduced by Mallat [4] is an increasing sequence $\{V_j\}_{j\in Z}$ of $L^2(R)$ such that $\cap_{j\in Z} V_j=\{0\}$, $\cup_{j\in Z} V_j$ is dense in $L^2(R)$ and $f(x)\in V_j$ iff $f(2x)\in V_{j+1}$. Furthermore there should exist an element $\phi \in V_0$ such that the collection of integer translates of ϕ forms an orthonormal system of V_0.. In the definition of multiresolution analysis the dilation factor of 2 can be replaced by an integer $N \geq 2$ and one can construct N-1 wavelets to generate the whole space $L^2(R)$. A similar generalization of multiresolution analysis can be made in higher dimensions by considering matrix dilations .

Gabardo and Nashed[5] considered a generalization of the notion of multiresolution analysis which is called nonuniform multiresolution analysis(NUMRA) and is based on the theory of spectral pairs. We have introduced Nonuniform multiresolution anlysis on positive half line in[6]. Xia and Suter [7] introduced vector-valued multiresolution analysis(VMRA) and orthogonal vector-valued wavelets. Sun and Cheng[8] presented an algorithm for construction of a class of compactly supported orthogonal vector-valued wavelets. Chen and Cheng[9] investigated the properties of vector-valued wavelet packets and constructed various orthonormal basis of $L^2(R;C^s)$ from the orthogonal vector-valued wavelet packets. We have considered a generalization of the notion of nonuniform multiresolution analysis(NUMRA) which is called vector-valued nonuniform multiresolution analysis(VNUMRA) in[10], where the associated subspace V_0 of $L^2(R;C^s)$ has, an orthonormal basis, a collection of translates of a vector valued function ϕ of the form $\{\phi(x-\lambda)\}_{\lambda\in\Lambda}$ where $\Lambda=\{0,r/N\}+2Z$, $N\geq 1$ is an integer and r is an odd integer with $1\leq r\leq 2N-1$ such that r and N are relatively prime Z and is the set of all integers and the corresponding dilation factor is 2N. We have obtained the necessary and sufficient condition for the existence of associated wavelets and presented a construction of vector-valued nonuniform multiresolution analysis. Recently we have studied the concept of Haar-Vilenkin wavelet in which is a generalization of Haar wavelet. We have introduced a Special Type of Multiresolution Analysis generated by Haar-Vilenkin wavelet [11] which is a special case of matrix multiresolution analysis studied in [12].

This paper is organized as follows: Vector-valued nonuniform multiresolution analysis(VNUMRA) is described in section 3 and a necessary and sufficient condition for the existence of associated wavelets is given. In section 4 vector-valued nonuniform wavelet packets are studied and their various properties are investigated. In section 5 the orthonormal basis of

$L^2(R;C^s)$ is constructed from vector-valued nonuniform wavelet packets and the orthonormal decomposition relation is reviewed.

3. Vector-Valued Nonuniform Multiresolution Analysis

Let s be a constant and $2 \le s \in Z$. By $L^2(R;C^s)$ we denote the set of all vector-valued functions $f(t)$ i.e.

$$L^2(R;C^s) = \{f(t) = (f_1(t), f_2(t), \dots, f_s(t))^T : t \in R, f_k(t) \in L^2(R), k = 1,2,\dots,s\}$$

(1)

Where ^{T} denotes the transpose and C^s denotes the s-dimensional complex Eucledian space.

Definition 1: Given integers $N \ge 1$ and r odd with $1 \le r \le 2N-1$ such that r and N are relatively prime and $\phi \in L^2(R,C^s)$ generates a VNUMRA $\{V_j\}_{j\in Z}$ of $L^2(R;C^s)$, if the sequence $\{V_j\}_{j\in Z}$ satisfies:

a) $\dots\dots \subset V_{-1} \subset V_0 \subset V_1 \subset \dots\dots$

b) $\cup_{j\in Z} V_j$ is dense in $L^2(R;C^s)$.

c) $\cap_{j\in Z} V_j = \{0\}$, where 0 is the zero vector of $L^2(R;C^s)$.

d) $\phi(t) \in V_j$ if and only if $\phi(2Nt) \in V_{j+1} \forall j \in Z.$

e) There exists $\phi(t) \in V_0$ such that the sequence $\{\phi(t-\lambda) : \lambda \in \Lambda\}$ is an orthonormal basis of V_0 where $\Lambda = \{0, r/N\} + 2Z$. The vector valued function $\phi(t)$ is called a scaling function of the VNUMRA.

Define a closed subspace $V_j \in L^2(R,C^s)$ by

$$V_j = clos_{L^2(R,C^s)}(span\{\phi((2N)^j t - \lambda), \lambda \in \Lambda\}), j \in Z .$$

(2)

Given a VNUMRA let W_m denotes the orthogonal complement of V_m in V_{m+1}, for any integer m. It is clear from the conditions (a), (b) and (c) of Definition 1 that

$$L^2(R,C^s) = \oplus_{m\in Z} W_m$$

The main purpose of VNUMRA is to construct orthonormal basis of $L^2(R;C^s)$ given by appropriate translates and dilates of finite collection of functions called the *associated wavelets.*

Definition 2: A collection $\{\psi_k\}_{k=1,2,\dots,2N-1}$ of functions in V_1 is called a set of wavelets associated with given VNUMRA if the family of functions $\{\psi_k(x-\lambda)\}_{k=1,2,\dots,2N-1,\lambda\in\Lambda}$ is an orthonormal system of W_0.

4. Vector-valued Nonuniform wavelet packets

We will define the vector-valued nonuniform wavelet packets(VNUWP) in this section and investigate their properties. Let

$$\gamma^0(t) = \phi(t), \gamma^k(t) = \psi_k(t), k = 1,2,\dots,2N-1$$

Definition 3: The family of vector-valued nonuniform functions $\{\gamma^{2Nn+k}(t), n \in Z_+, k = 0,1,\ldots$ …, 2N-1} is called a vector-valued nonuniform wavelet packet w.r.t the orthogonal vector-valued scaling function $\gamma^0(t)$, where

$$\gamma^{2Nn+k}(t) = \sum_{\lambda\in\Lambda} Q_\lambda^{(k)} \gamma^n(2Nt-\lambda), k = 0,1,\ldots,2N-1. \quad (3)$$

Note : Let $\{\psi_k(x-\lambda)\}_{k=1,2,\ldots,2N-1,\lambda\in\Lambda}$ is a system of functions orthonormal in V_1. Then the system is complete in $W_0 = V_1 - V_0$.

If $\psi_0, \psi_1, \ldots, \psi_{2N-1} \in V_1$ are as above, one can obtain from them an orthonormal basis for $L^2(R, C^s)$ by following the standard procedure for construction of wavelets from a given MRA [13], [14]. It can be easily checked that for every $m \in Z$, the collection $\{(2N)^{m/2}\psi_k((2N)^m x - \lambda)\}_{\lambda\in\Lambda, k=0,1,\ldots,2N-1}$ is a complete orthonormal system for V_{m+1} I.e. If $\{\gamma^n(t), n \in Z_+\}$ is a vector-valued nonuniform wavelet packet with respect to orthogonal vector-valued nonuniform scaling function $\phi(t)$, then $\forall n \in Z_+$ and $k,l \in \{0,1,\ldots 2N-1\}$

$$\left\langle \gamma^{2Nn+k}(.-\lambda), \gamma^{2Nn+l}(.-\sigma) \right\rangle = \delta_{\lambda,\sigma}\delta_{k,l} I_s, \lambda,\sigma \in \Lambda.$$

5. Vector-valued nonuniform wavelet bases

In this section we shall construct orthogonal vector-valued nonuniform wavelet bases of $L^2(R, C^s)$ by using orthogonal vector-valued nonuniform wavelet packets.

Define a dilation operator by (DF)(t) = F(2Nt) for $F \in L^2(R, C^s)$.

$\forall \Omega \subset L^2(R, C^s)$ and $\forall n \in Z_+$, denote $D\Omega = \{DF : F \in \Omega\}$ and

$$\Omega_n = \{F(t) : F(t) = \sum_{\lambda\in\Lambda} M_\lambda \gamma^n(t-\lambda), \{M_\lambda\}_{\lambda\in\Lambda} \in l^2(Z)^{s\times s}\}.$$

Then $\Omega_0 = V_0$ and $\Omega_1 \oplus \Omega_2 \oplus \Omega_3 \oplus \ldots \oplus \Omega_{2N-1} = W_0$, where $\oplus$ denotes the orthogonal direct sum.

Let $X, Y \subset R$ and $aX = \{ax : x \in X\}$ for $a \in R$.

$X + Y = \{x + y : x \in X, y \in Y\}, X - Y = \{x - y : x \in X, y \in Y\}.$

For a fixed positive integer r denote

$\widetilde{U}_r = \sum_{i=0}^{r} (2N)^i \{0,1,2,\ldots,2N-1\}$ and $U_r = \widetilde{U}_r \setminus \widetilde{U}_{r-1}$.

Note 1. For each $r \in Z_+ \setminus \{0\}$, the family of vector-valued functions $\{\gamma^n((2N)^j t - \lambda), n \in U_r, j \in Z, \lambda \in \Lambda\}$ forms an orthonormal basis of $L^2(R, C^s)$.

1. If $\{\gamma^n(t), n \in Z_+\}$ is a vector-valued nonuniform wavelet packet with respect to orthogonal vector-valued nonuniform scaling function $\phi(t)$, then $\forall n \in Z_+$ and $k,l \in \{0,1,\ldots 2N-1\}$

$$\left\langle \gamma^{2Nn+k}(.-\lambda), \gamma^{2Nn+l}(.-\sigma) \right\rangle = \delta_{\lambda,\sigma}\delta_{k,l} I_s, \lambda,\sigma \in \Lambda.$$

2. For any $n_1, n_2 \in Z_+$ and $\lambda, \sigma \in \Lambda$, we have

$$\left\langle \gamma^{n_1}(.-\lambda), \gamma^{n_2}(.-\sigma) \right\rangle = \delta_{n_1,n_2}\delta_{\lambda,\sigma} I_s,$$

Where $\{\gamma^n(t), n \in Z_+\}$ is VNUWP with respect to orthogonal vector-valued scaling function $\phi(t)$.

6. Conclusions

We have reviewed the properties of vector-valued non-uniform wavelets and wavelet packets. In future we will work on their applications in image processing, denoising and other areas.

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