Insight in Properties of Haar-Vilenkin Wavelets

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ABSTRACT *This paper deals with the study of Haar-Vilenkin wavelet which is also known as generalized Haar wavelets. Haar wavelet was introduced by Hungarian mathematician Alfred Haar. Haar-Vilenkin system was introduced around 1947. This system forms a wavelet system and we have proved the properties of this system. We have also introduced a Special Type of Multiresolution analysis generated by Haar-Vilenkin wavelet which is a special case of matrix multiresolution analysis. In this paper the different properties of Haar-Vilenkin wavelets and scaling function are discussed. Their representation in discrete form with the construction of Haar-Vilenkin matrices is also shown here and also expanded a function in Haar-Vilenkin wavelet series.*

Keywords: *Wavelets, Haar-Vilenkin system, scaling function, multiresolution analysis, matrices.*
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We have introduced the concept of Haar-Vilenkin wavelet and Haar-Vilenkin scaling function and have studied the basic properties of Haar-Vilenkin wavelet series and coefficients in [8]. Haar-Vilenkin wavelet is a generalization of Haar wavelet. Haar wavelet basis is the first example of an orthonormal wavelet basis. Haar basis functions are step functions with jump discontinuities [5]. Haar wavelet basis provides a very efficient representation of functions that consist of smooth, slowly varying segments punctuated by sharp peaks and discontinuities. Haar system is an orthonormal system such that each continuous function on [0, 1] has a uniformly convergent Fourier series with respect to this system.

The Haar function on real line R is defined as $h(x) = \begin{cases} 1, & x \in [0,1/2) \\ -1, & x \in [1/2,1) \\ 0, & \text{otherwise} \end{cases}$

The system $\{h_{j,k}(x)\}_{j,k\epsilon Z}$ where $h_{j,k}(x) = h(2^j x - k)$, $j, k \epsilon Z$, on taking the translations and dilations of $h(x)$, is defined as the Haar system on R . On the real line the Haar scaling function is $p(x) = \chi_{[0,1)}(x)$. The system of functions $\{p_{j,k}(x)\}_{j,k\epsilon Z}$, is referred to as the system of Haar scaling functions. The properties of Haar system have been studied. It has been proved that the system $\{p_{j,k}(x)\}_{j,k\epsilon Z}$ forms an orthonormal system in $L^2(R)$ [2, 17, 19]. The family $\{p_{j,k}\}_{j,k\epsilon Z}$ is also associated with multiresolution analysis.

Theorem 1.1 [19] The system $\{h_{j,k}(x)\}_{j,k\epsilon Z}$ forms an orthonormal basis in $L^2(R)$.

The convergence of the system in $L^p[0, 1]$ for $1 \leq p < \infty$ has been shown by Schauder in[12]. The comparison of Fourier series of a function $f \epsilon L^2(R)$ and its expansion with respect to the Haar system has been extensively studied. Behavior of Haar coefficients are also studied.

Behavior of Haar coefficients near jump discontinuities The following estimates are obtained in[17]. Suppose $f(x)$ is a function defined on [0, 1] with a jump discontinuity at $x\epsilon$ (0,1) and continuous at all other points in [0, 1]. Let us assume that the function $f(x)$ is C^2 on the intervals $[0, x_0]$ and $[x_0, 1]$. Now we have two possibilities, either $x_0 \epsilon \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right[$ or x_0 does not lie in $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right[$.

Case I If $x_0 \epsilon \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right[$, then, $|< f, p_{j,k} >| \approx \frac{1}{4} |f(x_0^-) - f(x_0^+)| 2^{-j/2}$.

Case II If x_0 does not lie in $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right[$, then $|< f, p_{j,k} >| \approx \frac{1}{4} |f'(x_{j,k})| 2^{-3j/2}$.

Thus we see that the decay of <f, $p_{j,k}$> for large j is considerably slower if $x_0 \epsilon \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right[$.
That is large coefficient in Haar expansion of a function $f(x)$ that persist for all scales suggest the presence of jump discontinuity in the intervals $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right[$ corresponding to the large coefficients.

It may be observed that Haar function was introduced in 1910 [5], Walsh function in 1923[18] and Haar type Vilenkin system in 1947, see for e.g.[16, 15, 13]. Certain properties of multi-dimensional generalized Haar type Fourier series have been investigated[14]. In the recent years various extensions and concepts related to Haar wavelet have been studied, see e.g.[1, 3, 4, 7, 14] . The matrix form of Haar wavelets, the integrals related to it and the solution of ODE's using Haar Wavelet coefficients is studied in [6]. In this chapter we have studied the properties of Haar-Vilenkin Wavelet series.
In this paper the concept of Haar-Vilenkin wavelets is recalled in section 2 and its various properties have been discussed. The concept of a special type of multiresolution analysis which generates Haar-Vilenkin wavelets is studied in section 3. Integrals related to Haar-Vilenkin wavelets have been evaluated in section 4 and wavelets have been represented in matrix form and general procedure for expanding a function or a signal in Haar-Vilenkin wavelet series is explained.

2 Haar-Vilenkin Wavelet

In this section we are recalling the system of Haar-Vilenkin wavelets studied in [8]:
The following system which is a generalization of Haar system is connected with the name of Vilenkin. Very often it is termed as a generalized Haar system or a Haar type Vilenkin system.
Let $m = (m_k, k \epsilon \mathrm{N})$ be a sequence of natural numbers such that $m_k \geq 2$, N denotes the set of non-negative integers. Let $M_0 = 1$ and $M_k = m_{k-1}M_{k-1}$, $k \epsilon$ P. Let P denotes the set of positive integers and let $k \epsilon$ P can be written as

$$k = M_n + r(m_n - 1) + s - 1, \tag{2.1}$$

where $n \epsilon \mathbf{N}$, $r = 0, 1, \ldots, M_n - 1$ and $s = 1, 2, \ldots, m_n - 1$. This expression is unique for each $k \epsilon \mathbf{P}$. Let us write an arbitrary element $t \epsilon [0,1)$ in the form

$$t = \sum_{k=0}^{\infty} \frac{t_k}{M_{k+1}}, (0 \leq t_k < m_k) \tag{2.2}$$

It may be noted that there exists two such expressions (2.2), for so called m- adic rational numbers. In such cases we use the expression which contains only a finite number of terms different from zero.

Define the function system ($h_k, k \epsilon \mathbf{N}$) by $h_0 = 1$ and
$$h_k(t) = \begin{cases} \sqrt{M_n} \exp \dfrac{2\pi i s t_n}{m_n} & \dfrac{r}{M_n} \leq t < \dfrac{r+1}{M_n} \\ 0 & otherwise \end{cases} \tag{2.3}$$

This system can be extended to R by periodicity of period 1: $h_k(t+1) = h_k(t)$, $t \epsilon [0, 1)$. It can be checked that $\{h_k(t)\}$ is a complete orthonormal system in $L^2(\mathrm{R})$.
Certain properties of this system have been recently studied [8]. For $k \epsilon$ P and $t \epsilon [0, 1)$ as defined in (2.1) and (2.2) the Haar -Vilenkin scaling function is defined as:

$$p_k(t) = \begin{cases} \sqrt{M_n}, & \dfrac{r}{M_n} \leq t < \dfrac{r+1}{M_n} \\ 0 & otherwise \end{cases} \tag{2.4}$$

The system $\{\phi_{a,b}(t)\}_{a,b \in Z}$ is Haar-Vilenkin scaling functions system where

$$\phi_{a,b}(t) = m_n^{a/2} p_k(m_n^a t - b).$$

We have studied the basic properties of Haar-Vilenkin system in [8]. We have proved the orthogonality of Haar-Vilenkin wavelet, convergence of Haar- Vilenkin wavelet series and properties of Haar-Vilenkin wavelet coefficients. We have introduced a multiresolution analysis where translation and dilation are taken by b/M_n where b lies in Z. The system $\{\psi_{a,b}(t)\}$ where a,b lie in Z is referred as Haar-Vilenkin system, where $\psi_{a,b}(t) = m^{a/2}h_k(m^a t - b)$.

Remark 2.1 1. Haar system is a special case of H a a r -Vilenkin system for $m_n = 2$ for all $n \in N$.
2. Given any $a \in Z$, the collection of scale a Haar-Vilenkin scaling functions is an orthonormal system on R.
3. For k=1, $h_k(t)$ is well known mother Haar wavelet.

Approximation Operator in Context of Haar-Vilenkin Wavelets

Definition 2.1 For each a lies in Z define the approximation operator P_a on the functions f in L^2(R) as $P_a f(x) = \sum_b < f, \emptyset_{a,\frac{b}{M_n}} > \emptyset_{a,\frac{b}{M_n}}$.
We have proved the following facts about the operator P_a in[8]: (2.5)

Theorem 2.1 1. For each $a \in Z$, P_a is linear, given f(x), g(x)$\in L^2$(R) and $\alpha, \beta \in C$
$P_a(\alpha f + \beta g)(x) = \alpha P_a(f) + \beta P_a(g)$.
2. For each $a \in Z$, P_a is idempotent.
3. For a, a' $\in$Z with $a \leq a'$ and for g(x) $\in V_a$ where $V_a = span\{\emptyset_{a,\frac{b}{M_n}}\}$, $b \in Z$ we have $P_{a'}g(x) = g(x)$.
4. Given $a \in Z$ and f(x) $\in L^2$(R) $||P_a f|| \leq ||f||_2$.
For f(x), C^0 in R we have $\lim_{a\to\infty} ||P_a f - f||_2 = 0$.

Theorem 2.2 [8] The system $\{\varphi_{a,b}\}_{a,b\in Z}$ forms an orthonormal basis in L^2(R).

Effect of Jump Discontinuities on the behavior of Coefficients of Haar-Vilenkin System
Consider a function f(x) on the interval $\left[\frac{r}{M_n}, \frac{r+1}{M_n}\right]$ with jump discontinuity at $x_0 \in \left(\frac{r}{M_n}, \frac{r+1}{M_n}\right)$ and continuous at all other points in $\left[\frac{r}{M_n}, \frac{r+1}{M_n}\right]$. We have to find the value of $< f, \psi_{a,b} >$ i.e. Haar-Vilenkin coefficients for $x_0 \in I_{a,b}$ the coefficients for x_0 *does not lies in* $I_{a,b}$.
Suppose that on the intervals $\left[\frac{r}{M_n}, x_0\right]$ and $\left[x_0, \frac{r+1}{M_n}\right]$, the function f(x) is C^2. Consider $x_{a,b}$ as the mid point of $I_{a,b}$ for the fix integers $a \geq 0$ and $0 \leq b \leq {m_n}^a - 1$.
Case I: If x_0 *does not lies in* $I_{a,b}$, then we have
$\left|< f, \psi_{a,b} >\right| \approx \frac{1}{4} {m_n}^{-3a/2} {M_n}^{-3/2} \left|f'(x_{a,b})\right|$, for large values of a.
Case II: If $x_0 \in I_{a,b}$, then if a is large, we have

$$\left|< f, \psi_{a,b} >\right| \approx {m_n}^{a/2} {M_n}^{1/2} \frac{1}{2{m_n}^a M_{n+1}} |f(x_0^-) - f(x_0^+)|$$
$$= \frac{{m_n}^{-a/2} {M_n}^{1/2}}{2M_{n+1}} |f(x_0^-) - f(x_0^+)|.$$

From the above two cases, we have the observation that for large values of a decay of $\left|< f, \psi_{a,b} >\right|$ is slower if $x_0 \in I_{a,b}$ rather than if x_0 *does not lies in* $I_{a,b}$.

3 A special type of Multiresolution Analysis

The results of this section are introduced in [9, 11]:
Definition 3.1 For k as in taken in (2.1), a special type of multiresolution analysis(MRA) is generated by a sequence which are closed subspaces $\{V_j\}_{j\in Z}$ of square integrable functions over R such that

1. $V_j \subset V_{j+1}$ for all integers j.
2. $\cup_{j\in Z} V_j$ is dense in L^2(R).
3. $\cap_{j\in Z} V_j$ is a zero subspace.

4. $f(x) \in V_j$ iff $f(m_n^{-j}x) \in V_0$ for all integers j.

5. There exists $g_k(x)$ in $L^2(R)$, such that the system of translates of g_k i.e. $\{g_k\left(t - \frac{b}{M_n}\right)\}_{b \in Z}$ is an orthonormal system, here $g_k(x)$ is called a scaling function and

$$V_0 = \overline{span}\left\{T_{\frac{b}{M_n}} g_k(x)\right\}.$$

Remark 3.1 In order to define a special type of MRA, first we identify the space V_0 and then take V_j as
$V_j = \{f(x) : f(x) = D_{m_n^j} g_k(x), g_k(x) \in V_0\}$
so that the Definition 3.1(4) is satisfied and then we prove that the conditions (1), (2),(3) and (5) of Definition 3.1 hold. First identify the function $g_k(x)$ and define V_0 such that the system of translates i.e. $\{T_{\frac{b}{M_n}} g_k(x)\}_{b \in Z}$ over Z is an orthonormal system and take

$$V_0 = \overline{span}\left\{T_{\frac{b}{M_n}} g_k(x)\right\}.$$

Example 3.1 [Haar-Vilenkin MRA] Consider the set V_0 of the function f(x) which are the step functions and satisfy

(i) f(x) is square integrable over R.

(ii) Over the interval $I_{0,\frac{b}{M_n}} = \left[\frac{r+b}{M_n}, \frac{r+b+1}{M_n}\right[$, f is constant $\forall b \in Z$.

It can be verified that for $l \in Z$, $V_0 = \overline{span}\left\{T_{\frac{l}{M_n}} g_k(x)\right\}$ where $p_k(x) = \sqrt{M_n}\,\chi_{\left[\frac{r}{M_n}, \frac{r+1}{M_n}\right)}$.

4 Integral forms of Haar-Vilenkin Wavelets

In this section we are introducing the results proved in [10].

Define the Haar-Vilenkin system $(h_k, k \in N)$ taken as over [A,B] of length 1 as h_0=1 and

$$h_k(t) = \begin{cases} \sqrt{M_n} \exp\frac{2\pi i s t_n}{m_n} & A + \frac{r}{M_n} \le t < A + \frac{r+1}{M_n} \\ 0 & otherwise \end{cases} \qquad (4.1)$$

for k as defined in 2.1. Let

$$P_{v,i}(x) = \int_A^x \int_A^x \dots \int_A^x h_i(t) dt^v$$
$$= \frac{1}{(v-1)!} \int_A^x (x-t)^{v-1} h_i(t) dt.$$

For $i \neq 1$ and for $\frac{r}{M_n} \le x < \frac{r}{M_n} + \frac{1}{M_{n-1}}$, we have

$$P_{\alpha,i}(x) = \frac{1}{(\alpha-1)!} \int x(x-t)^{\alpha-1} \sqrt{M_n} dt$$
$$= \frac{\sqrt{M_n}}{(\alpha-1)!} \int x(x-t)^{\alpha-1} dt$$
$$= \frac{\sqrt{M_n}}{(\alpha-1)!} \frac{1}{\alpha} \left(x - \frac{r}{M_n}\right)^{\alpha}$$
$$= \frac{\sqrt{M_n}}{\alpha!} \left(x - \frac{r}{M_n}\right)^{\alpha}$$

If $\frac{r}{M_n} + \frac{1}{M_{n-1}} \le x < \frac{r}{M_n} + \frac{2}{M_{n-1}}$, then on solving as above, we have

$$P_{\alpha,i}(x) = \frac{\sqrt{M_n}}{\alpha!} \left[x - \left(\frac{r}{M_n} + \frac{1}{M_{n-1}}\right)\right]^{\alpha} e^{2\pi i s / m_n}$$

…

…

If $\frac{r}{M_n}+\frac{m_n-1}{M_{n-1}} \le x < \frac{r+1}{M_n}$, we have

$$P_{\alpha,i}(x) = \frac{\sqrt{M_n}}{\alpha!}\left[x-\left(\frac{r}{M_n}+\frac{m_n-1}{M_{n-1}}\right)\right]^{\alpha} e^{2\pi is(m_n-1)/m_n}.$$

Thus

$$P_{\alpha,i}(x) = \begin{cases} 0 & x < \frac{r}{M_n} \\ \frac{\sqrt{M_n}}{\alpha!}\left(x-\frac{r}{M_n}\right)^{\alpha} & \frac{r}{M_n} \le x < \frac{r}{M_n}+\frac{1}{M_{n-1}} \\ \frac{\sqrt{M_n}}{\alpha!}\left[x-\left(\frac{r}{M_n}+\frac{1}{M_{n-1}}\right)\right]^{\alpha} e^{2\pi is/m_n} & \frac{r}{M_n}+\frac{1}{M_{n-1}} \le x < \frac{r}{M_n}+\frac{2}{M_{n-1}} \\ \dots & \dots \\ \frac{\sqrt{M_n}}{\alpha!}\left[x-\left(\frac{r}{M_n}+\frac{m_n-1}{M_{n-1}}\right)\right]^{\alpha} e^{2\pi is(m_n-1)/m_n} & \frac{r}{M_n}+\frac{m_n-1}{M_{n-1}} \le x < \frac{r+1}{M_n} \end{cases} \quad (4.2)$$

We have $h_i(t) = 0$ for i=0 and

$$P_{\alpha,1}(x) = \frac{1}{(\alpha-1)!}\int_A^x (x-t)^{\alpha-1}\,dt. = \frac{1}{(\alpha-1)!}\frac{1}{\alpha}(x-A)^{\alpha} = \frac{(x-A)^{\alpha}}{(\alpha)!} \quad (4.3)$$

Thus we have equation (4.2) for $i > 1$ and (4.3) for i=1.

4.1 Haar-Vilenkin Wavelets in Matrix Form

We have constructed the wavelets in discrete form for the case where A=0 and B=1. The grid points are denoted as

$$\widetilde{x_l} = A + l\,\delta x,\ \ l = 0,1,2,\dots,m_0. \quad (4.4)$$

We have considered

$$x_l = \frac{1}{2}(\widetilde{x_{l-1}} + \widetilde{x_l}),\ l = 1,2,\dots,m_0 \quad (4.5)$$

We get the Haar-Vilenkin wavelets on replacing x by x_l in equations (4.1), (4.2) and (4.3). The elements of square matrices H, P_1, P_2, …, P_v, that we have introduced are

$$H(i,l) = h_i(x_l), S_v(i,l) = P_{v_i}(x_l), v = 1,2,3 \dots$$

A=0, B=1 and

$$\delta x = \frac{B-A}{m_0} = \frac{1}{m_0}.$$

Example 4.2 Take a sequence $(m_k, k \in N) = (2,2,2,\dots)$.
Then $x_1=1/4$ and $x_3=3/4$. Then

$$h_1(t) = \begin{cases} 1 & 0 \le t < 1/2 \\ -1 & \frac{1}{2} \le t < 1 \\ 0 & otherwise \end{cases}$$

$$h_1(t) = \begin{cases} \sqrt{2} & 0 \le t < 1/4 \\ -\sqrt{2} & \frac{1}{4} \le t < 1/2 \\ 0 & otherwise \end{cases}$$

The Haar-Vilenkin matrix is formulated as

$$H = \begin{bmatrix} h_1(x_1) & h_1(x_2) \\ h_2(x_1) & h_2(x_2) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ \sqrt{2} & 0 \end{bmatrix}.$$

The other matrices are

$$S_1 = \begin{bmatrix} P_{11}(x_1) & P_{11}(x_2) \\ P_{12}(x_1) & P_{12}(x_2) \end{bmatrix}, S_2 = \begin{bmatrix} P_{21}(x_1) & P_{21}(x_2) \\ P_{22}(x_1) & P_{22}(x_2) \end{bmatrix} \dots$$

Using the equations (4.4) and (4.5), we have

$$P_1 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{32} & \frac{9}{32} \end{bmatrix}.$$

4.2 Exanpsion of Function in Haar-Vilenkin Wavelet Series

Let f ϵ $L^2[A,B]$. It can be expanded in Haar-Vilenkin wavelet series as

$$f(x) = \sum_{i=1}^{m_0} a_i h_i(x), \quad (4.6)$$

where a_i denotes the Haar-Vilenkin wavelet coefficients. The discrete form of this expansion $\hat{f}(x_l) = \sum_{i=1}^{m_0} a_i h_i(x_l)$ (4.7)

The matrix form of (4.7) is f= a H, where H is Haar-Vilenkin matrix.
Both a = (a_i) and f = (f_i) are m_0 dimensional row vectors. We obtain $a = fH^{-1}$.
On replacing the value of a in (4.6), we obtain wavelet approximation of f.

Conclusion

In this paper we have reviewed the basic properties of Haar wavelets. Haar-Vilenkin wavelets which are also termed as generalized Haar wavelets and its various properties are studied. A special type of multiresolution analysis is studied. The Haar-Vilenkin wavelets have also been studied in integral and matrix form. These methods will be useful in solution of ordinary and partial differential equations.

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