On Certain Subclasses of Multivalent Functions with Negative Coefficients and (m,q)-Starlike with Respect to Certain Points

Manisha Summerwar
Lecturer
Govt. Girls Sr. Sec. School
Kishangarh, Ajmer (Raj.), INDIA

S. K. Bissu
Department of Mathematics
S.P.C. Govt. College, Ajmer
Ajmer (Raj.), INDIA

Received August 10, 2017 Accepted Sept. 05, 2017

ABSTRACT *In this Paper we have introduced and study subclasses of multivalent functions with negative coefficient by using Dziok-Srivastava operator defined in unit disk. Here we have studied coefficient estimates, Distortion Theorems, Extreme Points. These results include many results as particular cases.*
Key words: *p-valent functions, Coefficient Estimates, Distortion Bounds, Extreme Points*

1. INTRODUCTION

Let C_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} \alpha_{k+p} z^{k+p} \tag{1}$$

which are analytic and multivalent in the open unit disk U = {z: $|z| < 1$} and normalized by f' (0) = f (0) +1 = 1. Let f ∈ C_p given by (1) and g ∈ C_p given by

$$g(z) = z^p + \sum_{k=1}^{\infty} \beta_{k+p} z^{k+p} \quad . \tag{2}$$

We define the convolution product (or Hadamard) of f and g by

$$(f*g)(z) = z^p + \sum_{k=1}^{\infty} \alpha_{k+p}\ \beta_{k+p}\, z^{k+p} = (g*f)(z); \qquad (z \in U) \tag{3}$$

For +ve real parameters a_1, a_2, -----, a_m and b_1, b_2, ------, b_q .
$\left(b_j \in \mathbb{C}/\mathbb{Z}_0^-, \mathbb{Z}_0^- = 0, -1, -2, \dots\dots\dots; j = 1,2, \dots\dots, q\right)$, the generalized hyper geometric function ${}_mF_q$ (a_1, a_2, -----------, a_m, b_1, b_2, -------------, b_q ; z) is defined by

$${}_mF_q\ (a_1, a_2, \dots\dots\dots\dots\dots\, a_m; b_1, b_2, \dots\dots\dots\dots\, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots\dots\dots.(a_m)_n}{(b_1)_n \dots\dots\dots.(b_q)_n} . \frac{1}{n!} z^n$$

$(q \le s+1\,; s, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1,2, \dots.\};$ z ∈ U) Where$(\ \ )_n$ is the Pochhammer symbol defined in terms of the Gamma function, by

$$(\ \ )_n = \frac{\Gamma(\phi + n)}{\Gamma(\phi)} = \begin{cases} 1 & , n = 0 \\ \phi(\phi+1) \dots\dots (\phi + n - 1) & , n \in \mathbb{N} \end{cases}$$

For the function

h (a_1, a_2, ---, a_m, b_1, b_2, ----, b_q ; z) =z ${}_qF_s$ (a_1, a_2, ----, a_m, b_1, b_2, ------, b_q ; z),

the Dziok-Srivastava linear operator (see [1]).

$H_{m,q}$ (a_1, a_2, ---------, a_m, b_1, b_2, --------, b_q) : $C_p \to C_p$

is defined by the Hadamard product as follows:

$H_{m,q}$ (a_1, a_2, ----, a_m, b_1, b_2, ----, b_q) f(z) = h (a_1, a_2, ---, a_m, b_1, b_2, ----, b_q ; z)*f(z)

$$= z^p + \sum_{n=1}^{\infty} \Gamma_n(a_1)\alpha_{n+p}\, z^{n+p} \qquad (z \in U) \tag{4}$$

Where

$$\Gamma_n(a_1) = \frac{(a_1)_{n-1} \dots\dots\dots (a_m)_{n-1}}{(b_1)_{n-1} \dots\dots\dots (b_q)_{n-1}} . \frac{1}{(n-1)!} \tag{5}$$

For brevity, we write

$H_{m,q}$ (a_1, a_2, ----, a_m, b_1, b_2, ----, b_q; z) f(z) = $H_{m,q}$ (a_1) f(z)

Definition 1. Let the function f (z) be defined by (1) then $f(z) \in S_{m,q}(z)$ if and only if

$$Re\ \frac{z\left(H_{m,q}\,(a_1)f(z)\right)'}{H_{m,q}\,(a_1)f(z)} > 0. \tag{6}$$

Definition 2. Let the function f (z) be defined by (1) then $f(z) \in S_{m,q}^*(z)$ if and only if

$$Re\ \frac{z\left(H_{m,q}\,(a_1)f(z)\right)'}{H_{m,q}\,(a_1)f(z) - H_{m,q}\,(a_1)f(-z)} > 0, \tag{7}$$

these classes of functions are called starlike with respect to symmetric points in U.
Let T_p denotes the subclasses of C_p consisting of functions of the form :

$$f(z) = z^p - \sum_{k=1}^{\infty} \alpha_{k+p} z^{k+p} \qquad (\alpha_{k+p} \geq 0.) \tag{8}$$

Definition 3. Let the function f(z) be defined by (8). Then f(z) is said to be m-q starlike with respect to symmetric points if it satisfies the following condition:

$$\left| \frac{z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z) - H_{m,q}(a_1)f(-z)} - p \right| < \beta \left| \frac{\alpha\, z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z) - H_{m,q}(a_1)f(-z)} + p \right| \tag{9}$$

For some $0 < \beta \leq 1, 0 \leq \alpha \leq 1$ *and* $z \in U$. We denote the class m-q-starlike with respect to symmetric points by $S^*_{s,m,q}\, T_p(z)$.

Definition 4. Let the function f(z) be defined by (8). Then f(z) is said to be m-q starlike with respect to conjugate points if it satisfies the following condition:

$$\left| \frac{z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z) + \overline{H_{m,q}(a_1)f(\bar{z})}} - p \right| < \beta \left| \frac{\alpha\, z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z) - \overline{H_{m,q}(a_1)f(\bar{z})}} + p \right| \tag{10}$$

For some $0 < \beta \leq 1, 0 \leq \alpha \leq 1$ *and* $z \in U$. We denote the class m-q-starlike with respect to conjugate points by $S^*_{c,m,q}\, T_p(z)$.

Definition 5. Let the function f(z) be defined by (8). Then f(z) is said to be m-q starlike with respect to symmetric conjugate points if it satisfies the following condition:

$$\left| \frac{z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z) - \overline{H_{m,q}(a_1)f(\overline{-z})}} - p \right| < \beta \left| \frac{\alpha\, z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z) + \overline{H_{m,q}(a_1)f(\overline{-z})}} + p \right| \tag{11}$$

For some $0 < \beta \leq 1, 0 \leq \alpha \leq 1$ *and* $z \in U$. We denote the class m-q-starlike with respect to symmetric conjugate points by $S^*_{sc,m,q}\, T_p(z)$.

2. Coefficient Estimates

Theorem 1. Let the function f be defined by (8) and $H_{m,q}(a_1)f(z) - H_{m,q}(a_1)f(-z) \neq 0$ for z ≠ 0. Then f(z) is in the class $S^*_{s,m,q}\, T_p(z)$ if and only if

$$\sum_{k=1}^{\infty} \Gamma_k(a_1)\left[k(1+\beta\alpha) + p\left((-1)^{k+p} + \beta(\alpha + 1 - (-1)^{k+p})\right)\right]\alpha_{k+p} < \beta(\alpha + 1 - (-1)^p)p + p(-1)^p. \tag{12}$$

Proof: Let $f(z) \in S^*_{c,m,q}\, T_p(z)$, then

$$\left| \frac{z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z) - H_{m,q}(a_1)f(-z)} - p \right| < \beta \left| \frac{\alpha\, z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z) - H_{m,q}(a_1)f(-z)} + p \right|$$

Using (4), that is

$$H_{m,q}(a_1)f(z) = z^p - \sum_{k=1}^{\infty} \Gamma_k(a_1)\alpha_{k+p}\, z^{k+p}$$

and

$$H_{m,q}(a_1)f(-z) = (-1)^p z^p - \sum_{k=1}^{\infty} \Gamma_k(a_1)\alpha_{k+p}(-1)^{k+p} z^{k+p}$$

then we have

$$\left| z\left(H_{m,q}(a_1)f(z)\right)' - p\, H_{m,q}(a_1)f(z) + pH_{m,q}(a_1)f(-z) \right|$$
$$< \beta \left| z\left(H_{m,q}(a_1)f(z)\right)' + p\, H_{m,q}(a_1)f(z) - pH_{m,q}(a_1)f(-z) \right|$$

that is

$$\left| p\, z^p - \sum_{k=2}^{\infty} (p+k)\Gamma_k(a_1)\alpha_{k+p}\, z^{k+p} - p\, z^p + \sum_{k=2}^{\infty} p\Gamma_k(a_1)\alpha_{k+p}\, z^{k+p} + p\,(-1)^p z^p - p\sum_{k=1}^{\infty} \Gamma_k(a_1)\alpha_{k+p}(-1)^{k+p} z^{k+}\right.$$
$$< \beta \left| p\,\alpha\, z^p - \sum_{k=2}^{\infty} \alpha(p+k)\Gamma_k(a_1)\alpha_{k+p}\, z^{k+p} + p\, z^p - \sum_{k=2}^{\infty} p\Gamma_k(a_1)\alpha_{k+p}\, z^{k+p} - p\,(-1)^p z^p \right.$$
$$\left. + p\sum_{k=1}^{\infty} \Gamma_k(a_1)\alpha_{k+p}(-1)^{k+p} z^{k+p} \right|$$

Which readily gives

$$\left|p(-1)^p z^p - \sum_{k=1}^{\infty} \Gamma_k(a_1)\alpha_{k+p}\, z^{k+p}[k+p-p+p(-1)^{k+p}]\right| < \beta\left|(\alpha+1-(-1)^p)p\, z^p - \sum_{k=1}^{\infty} \Gamma_k(a_1) z^{k+p}\,[\alpha(k+p)+p-p(-1)^{k+p}]\right|$$

also,

$$|p(-1)^p z^p| + \sum_{k=1}^{\infty} \Gamma_k(a_1)\alpha_{k+p}\, |z|^{k+p}[k+p(-1)^{k+p}] < \beta\,|(\alpha+1-(-1)^p)p\, z^p| - \sum_{k=1}^{\infty} \Gamma_k(a_1)|z|^{k+p}\,[\alpha\beta(k+p)+\beta p-p\beta(-1)^{k+p}]$$

that is

$$-p(-1)^p|z|^p + \sum_{k=1}^{\infty} \Gamma_k(a_1)\alpha_{k+p}\, |z|^{k+p}[k+p(-1)^{k+p}] + \sum_{k=1}^{\infty} \Gamma_k(a_1)|z|^{k+p}\,[\alpha\beta(k+p)+\beta p-p\beta(-1)^{k+p}] < \beta\,(\alpha+1-(-1)^p)p|z|^p$$

Which gives

$$\sum_{k=1}^{\infty} \Gamma_k(a_1)\left[k(1+\beta\alpha)+\, p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right]\alpha_{k+p}\, |z|^p < [\beta\,(\alpha+1-(-1)^p)p+\, p(-1)^p]|z|^p$$

Let $|z| \to 1^-$. So we have

$$\sum_{k=1}^{\infty} \Gamma_k(a_1)\left[k(1+\beta\alpha)+\, p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right]\alpha_{k+p} < \beta(\alpha+1-(-1)^p)p+p\,(-1)^p.$$

Hence by maximum modulus theore, we have $f(z) \in S^*_{c,m,q}\,T_p(z)$.

Conversely: - we assume that

$$\left|\frac{\dfrac{z\left(H_{m,q}\,(a_1)f(z)\right)'}{H_{m,q}\,(a_1)f(z)-H_{m,q}\,(a_1)f(-z)}-p}{\dfrac{\alpha\, z\left(H_{m,q}\,(a_1)f(z)\right)'}{H_{m,q}\,(a_1)f(z)-H_{m,q}\,(a_1)f(-z)}+p}\right| < \beta$$

that is

$$\left|\frac{p(-1)^p z^p - \sum_{k=1}^{\infty}\Gamma_k(a_1)\alpha_{k+p}\, z^{k+p}[k+p-p+p(-1)^{k+p}]}{(\alpha+1-(-1)^p)p\, z^p - \sum_{k=1}^{\infty}\Gamma_k(a_1)z^{k+p}\,[\alpha(k+p)+p-p(-1)^{k+p}]}\right| < \beta$$

Now since $Re\, f(z) \le |f(z)|$ for all z we have

$$Re\left\{\frac{p(-1)^p z^p - \sum_{k=1}^{\infty}\Gamma_k(a_1)\alpha_{k+p}\, z^{k+p}[k+p-p+p(-1)^{k+p}]}{(\alpha+1-(-1)^p)p\, z^p - \sum_{k=1}^{\infty}\Gamma_k(a_1)z^{k+p}\,[\alpha(k+p)+p-p(-1)^{k+p}]}\right\} < \beta \tag{13}$$

On the real axis choose values of z, we have $\frac{z\left(H_{m,q}\,(a_1)f(z)\right)'}{H_{m,q}\,(a_1)f(z)-H_{m,q}\,(a_1)f(-z)} - p$ is real and $H_{m,q}\,(a_1)f(z) - H_{m,q}\,(a_1)f(-z) \neq 0$ for z ≠ 0. Upon clearing denominator in (13) and letting z →1⁻ along the real values, we

$$-p(-1)^p + \sum_{k=1}^{\infty}\Gamma_k(a_1)\alpha_{k+p}\,[k+p(-1)^{k+p}] < \beta\,(\alpha+1-(-1)^p)p - \sum_{k=1}^{\infty}\Gamma_k(a_1)\,[\alpha\beta(k+p)+\beta p-p\beta(-1)^{k+p}]\,\alpha_{k+p}$$

This completes the proof of Theorem 1. ▪

Corollary 1: Let the function f(z) defined by (8) be in the class $S^*_{s,m,q}\,T_p(z)$. Then we have

$$\alpha_{k+p} \le \frac{\beta(\alpha+1-(-1)^p)p+p\,(-1)^p}{\Gamma_k(a_1)\left[k(1+\beta\alpha)+p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right]} \quad (k \ge 1). \tag{14}$$

The equality in (14) is attained for the function f(z) given by

$$f(z) = z^p - \frac{\beta(\alpha+1-(-1)^p)p+p\,(-1)^p}{\Gamma_k(a_1)\left[k(1+\beta\alpha)+p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right]}\, z^{k+p} \quad (k \ge 1). \tag{15}$$

Theorem 2: Let the function f be defined by (8) and $H_{m,q}(a_1)f(z) - \overline{H_{m,q}(a_1)f(\bar{z})} \neq 0$ for z ≠ 0. Then f(z) is in the class $S^*_{c,m,q}T_p(z)$ if and only if

$$\sum_{k=1}^{\infty}\Gamma_k(a_1)[k(1+\beta\alpha) - p(1-\beta(\alpha+2))]\alpha_{k+p} < p\beta(\alpha+2) - p. \quad (16)$$

Corollary 2: Let the function f(z) defined by (8) be in the class $S^*_{c,m,q}T_p(z)$. Then we have

$$\alpha_{k+p} \leq \frac{p\beta(\alpha+2)-p}{\Gamma_k(a_1)[k(1+\beta\alpha)-p(1-\beta(\alpha+2))]} \quad (k \geq 1). \quad (17)$$

The equality in (17) is attained for the function f(z) given by

$$f(z) = z^p - \frac{p\beta(\alpha+2)-p}{\Gamma_k(a_1)[k(1+\beta\alpha)-p(1-\beta(\alpha+2))]} z^{k+p} \quad (k \geq 1). \quad (18)$$

Theorem 3. Let the function f be defined by (8) and $H_{m,q}(a_1)f(z) - \overline{H_{m,q}(a_1)f(\overline{-z})} \neq 0$ for z ≠ 0. Then f(z) is in the class $S^*_{sc,m,q}T_p(z)$ if and only if

$$\sum_{k=1}^{\infty}\Gamma_k(a_1)\left[k(1+\beta\alpha) + p\left((-1)^{k+p} + \beta(\alpha+1-(-1)^{k+p})\right)\right]\alpha_{k+p} < \beta(\alpha+1-(-1)^p)p + p(-1)^p. \quad (19)$$

Corollary 3: Let the function f(z) defined by (8) be in the class $S^*_{sc,m,q}T_p(z)$. Then we have

$$\alpha_{k+p} \leq \frac{\beta(\alpha+1-(-1)^p)p+p(-1)^p}{\Gamma_k(a_1)\left[k(1+\beta\alpha)+p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right]} \quad (k \geq 1). \quad (20)$$

The equality in (20) is attained for the function f(z) given by

$$f(z) = z^p - \frac{\beta(\alpha+1-(-1)^p)p+p(-1)^p}{\Gamma_k(a_1)\left[k(1+\beta\alpha)+p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right]} z^{k+p} \quad (k \geq 1). \quad (21)$$

3. Growth and Distortion Bounds

Theorem 4. Let the function f defined by (8) be in the class $S^*_{s,m,q}T_p(z)$. Then for $|z| = r < 1$, we have

$$|f(z)| \geq r^p - \frac{\beta(\alpha+1-(-1)^p)p+p(-1)^p}{(1+\alpha\beta)+p[(-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})]}.r^{p+1} \quad (22)$$

and

$$|f(z)| \leq r^p + \frac{\beta(\alpha+1-(-1)^p)p+p(-1)^p}{(1+\alpha\beta)+p[(-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})]}.r^{p+1} \quad (23)$$

for $z \in U$. The equalities in (22) and (23) are attained for the function given by

$$|f(z)| = z^p - \frac{\beta(\alpha+1-(-1)^p)p+p(-1)^p}{(1+\alpha\beta)+p[(-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})]}.z^{p+1} \quad (24)$$

at z = r .

Proof:- Since for k ≥ 1

$$(1+\alpha\beta) + p[(-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})] \leq \Gamma_k(a_1)\left[k(1+\beta\alpha) + p\left((-1)^{k+p} + \beta(\alpha+1-(-1)^{k+p})\right)\right]$$

using Theorem 1, we have

$$\{(1+\alpha\beta) + p[(-1)^{p+1} + \beta(\alpha+1-(-1)^{p+1})]\}\sum_{k=1}^{\infty}\alpha_{k+p}$$
$$\leq \sum_{k=1}^{\infty}\Gamma_k(a_1)\left[k(1+\beta\alpha) + p\left((-1)^{k+p} + \beta(\alpha+1-(-1)^{k+p})\right)\right]\alpha_{k+p}$$
$$\leq \beta(\alpha+1-(-1)^p)p + p(-1)^p$$

then

$$\sum_{k=1}^{\infty}\alpha_{k+p} \leq \frac{\beta(\alpha+1-(-1)^p)p+p(-1)^p}{(1+\alpha\beta)+p[(-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})]} \quad (25)$$

From (25) and (8), we have

$$|f(z)| \geq r^p - r^{p+1}\sum_{k=1}^{\infty}\alpha_{k+p} \geq r^p - \frac{\beta(\alpha+1-(-1)^p)p+p(-1)^p}{(1+\alpha\beta)+p[(-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})]} r^{p+1}$$

and

$$|f(z)| \leq r^p + r^{p+1}\sum_{k=1}^{\infty}\alpha_{k+p} \leq r^p + \frac{\beta(\alpha+1-(-1)^p)p+p(-1)^p}{(1+\alpha\beta)+p[(-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})]} r^{p+1}$$

This completes the proof of Theorem 4. ▪

Theorem 5:- Let the function f(z) defined by (8) be in the $S^*_{s,m,q}\,T_p(z)$. Then for $|z| = r < 1$, we have

$$|f'(z)| \geq p\, r^{p-1} - \frac{(\beta(\alpha+1-(-1)^p)p+p\,(-1)^p)(1+p)}{(1+\alpha\beta)+p[(-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})]}\, r^p \tag{26}$$

and

$$|f'(z)| \leq p\, r^{p-1} - \frac{(\beta(\alpha+1-(-1)^p)p+p\,(-1)^p)(1+p)}{(1+\alpha\beta)+p[(-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})]}\, r^p \tag{27}$$

for $z \in U$. The equalities in (22) and (23) are attained for the function f(z) given by (24)

.

Proof:- for k ≥ 1. Using theorem 1, we have

$$\{(1+\alpha\beta)+p[(-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})]\}\sum_{k=1}^{\infty}(k+p)\alpha_{k+p}$$

$$\leq (1+p)\sum_{k=1}^{\infty}\Gamma_k(a_1)\left[k(1+\beta\alpha)+p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right]\alpha_{k+p}$$

$$\leq (1+p)\{\beta(\alpha+1-(-1)^p)p+p\,(-1)^p\}$$

then

$$\sum_{k=1}^{\infty}(k+p)\alpha_{k+p} \leq \frac{(\beta(\alpha+1-(-1)^p)p+p\,(-1)^p)(1+p)}{(1+\alpha\beta)+p[(-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})]} \tag{28}$$

From (28) & (8), we have

$$|f'(z)| \geq p\,r^{p-1} - r^p\sum_{k=1}^{\infty}(k+p)\alpha_{k+p} \geq p\,r^{p-1} - \frac{(\beta(\alpha+1-(-1)^p)p+p\,(-1)^p)(1+p)}{(1+\alpha\beta)+p[(-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})]}\, r^p$$

And

$$|f'(z)| \leq p\,r^{p-1} + r^p\sum_{k=1}^{\infty}(k+p)\alpha_{k+p} \leq p\,r^{p-1} + \frac{(\beta(\alpha+1-(-1)^p)p+p\,(-1)^p)(1+p)}{(1+\alpha\beta)+p[(-1)^{p+1}+\beta(\alpha+1-(-1)^{p+1})]}\, r^p$$

This completes the proof of Theorem 5. ▪

On similar lines of Theorem 4 and Theorem 5, we can easily prove the following Theorem 6 and Theorem 7 respectively for f(z) belongs to $S^*_{c,m,q}\,T_p(z)$.

Theorem 6. Let the function f defined by (8) be in the class $S^*_{c,m,q}\,T_p(z)$. Then for $|z| = r < 1$, we have

$$|f(z)| \geq r^p - \frac{p\beta(\alpha+2)-p}{(1+\alpha\beta)-p\,[1-\beta(\alpha+2)]}.\,r^{p+1} \tag{29}$$

and

$$|f(z)| \leq r^p + \frac{p\beta(\alpha+2)-p}{(1+\alpha\beta)-p\,[1-\beta(\alpha+2)]}.\,r^{p+1} \tag{30}$$

for $z \in U$. The equalities in (29) and (30) are attained for the function given by

$$|f(z)| = z^p - \frac{p\beta(\alpha+2)-p}{(1+\alpha\beta)-p\,[1-\beta(\alpha+2)]}.\,z^{p+1} \tag{31}$$

at z = r .

Theorem 7 :- Let the function f(z) defined by (8) be in the $S^*_{c,m,q}\,T_p(z)$. Then for $|z| = r < 1$, we have

$$|f'(z)| \geq p\,r^{p-1} - \frac{[p\beta(\alpha+2)-p](1+p)}{(1+\alpha\beta)-p\,[1-\beta(\alpha+2)]}\, r^p \tag{32}$$

and

$$|f'(z)| \leq p\,r^{p-1} - \frac{[p\beta(\alpha+2)-p](1+p)}{(1+\alpha\beta)-p\,[1-\beta(\alpha+2)]}\, r^p \tag{33}$$

for $z \in U$. The equalities in (32) and (33) are attained for the function f(z) given by (31).

4. Extreme Points

Theorem 8:- Let $f_p(z) = z^p$ and

$$f_{k+p}(z) = z^p - \frac{\beta(\alpha+1-(-1)^p)p+p\,(-1)^p}{\Gamma_k(a_1)\left[k(1+\beta\alpha)+p\left((-1)^{k+p}+\beta\left(\alpha+1-(-1)^{k+p}\right)\right)\right]}\, z^{k+p} \tag{34}$$

where k ≥ 1. Then $f(z) \in S^*_{s,m,q}\,T_p(z)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty}\Upsilon_{k+p}\, f_{k+p}(z) \tag{35}$$

Where $\Upsilon_{k+p} \geq 0$ (k ≥ 1) and $\sum_{k=0}^{\infty}\Upsilon_{k+p} = 1$.

Proof: Suppose

$$f(z) = \sum_{k=0}^{\infty} \Upsilon_{k+p}\, f_{k+p}(z)$$
$$= z^p - \sum_{k=0}^{\infty} \frac{\beta(\alpha+1-(-1)^p)p + p\,(-1)^p}{\Gamma_k(a_1)\left[k(1+\beta\alpha) + \ p\left((-1)^{k+p} + \beta(\alpha+1-(-1)^{k+p})\right)\right]}\, \Upsilon_{k+p}\, z^{k+p}$$

Then we get

$$\sum_{k=1}^{\infty} \frac{\beta(\alpha+1-(-1)^p)p + p\,(-1)^p}{\Gamma_k(a_1)\left[k(1+\beta\alpha) + \ p\left((-1)^{k+p} + \beta(\alpha+1-(-1)^{k+p})\right)\right]} \cdot \frac{\Gamma_k(a_1)\left[k(1+\beta\alpha) + \ p\left((-1)^{k+p} + \beta(\alpha+1-(-1)^{k+p})\right)\right]}{\beta(\alpha+1-(-1)^p)p + p\,(-1)^p}$$
$$= \sum_{k=1}^{\infty} \Upsilon_{k+p} = 1 - \Upsilon_p \le 1.$$

it follows from Theorem 1 that the function $f(z) \in S^*_{s,m,q}\, T_p(z)$.

Conversely: suppose that $f(z) \in S^*_{s,m,q}\, T_p(z)$. Again by using Theorem 1, we can show that

$$|\alpha_{k+p}| \le \frac{\beta(\alpha+1-(-1)^p)p+p\,(-1)^p}{\Gamma_k(a_1)\left[k(1+\beta\alpha)+ p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right]} \quad (k \ge 1). \tag{36}$$

setting

$$|\Upsilon_{k+p}| \le \frac{\Gamma_k(a_1)\left[k(1+\beta\alpha)+ p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right]}{\beta(\alpha+1-(-1)^p)p+p\,(-1)^p} \tag{37}$$

and

$$\Upsilon_p = 1 - \sum_{k=1}^{\infty} \Upsilon_{k+p} \tag{38}$$

We can see that f(z) can be expressed in the form of (34). This completes the proof of Theorem 8.

Corollary 4: The extreme points of the class $S^*_{s,m,q}\, T_p(z)$ are functions $f_{k+p}(z)$ ($k \ge 1, p \in N$) given by Theorem 8.

Similar to Theorem 8, we can easily prove the following theorems for $f(z) \in S^*_{c,m,q}\, T_p(z)$ and $f(z) \in S^*_{sc,m,q}\, T_p(z)$ classes.

Theorem 9:- Theorem 8:- Let $f_p(z) = z^p$ and

$$f_{k+p}(z) = z^p - \frac{p\beta(\alpha+2)-p}{\Gamma_k(a_1)[k(1+\beta\alpha)- p\,(1-\beta(\alpha+2))]}\, z^{k+p} \tag{39}$$

where k ≥ 1. Then $f(z) \in S^*_{s,m,q}\, T_p(z)$ if and only if it can be expressed in the form $f(z) = \sum_{k=0}^{\infty} \Upsilon_{k+p}\, f_{k+p}(z)$ where $\Upsilon_{k+p} \ge 0$ ($k \ge 1$) and $\sum_{k=0}^{\infty} \Upsilon_{k+p} = 1$.

Corollary 5: The extreme points of the class $S^*_{c,m,q}\, T_p(z)$ are functions $f_{k+p}(z)$ ($k \ge 1, p \in N$) given by Theorem 9.

Theorem 10:- Let $f_p(z) = z^p$ and

$$f_{k+p}(z) = z^p - \frac{\beta(\alpha+1-(-1)^p)p+p\,(-1)^p}{\Gamma_k(a_1)\left[k(1+\beta\alpha)+ p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right]}\, z^{k+p} \tag{40}$$

where k ≥ 1. Then $f(z) \in S^*_{sc,m,q}\, T_p(z)$ if and only if it can be expressed in the form $f(z) = \sum_{k=0}^{\infty} \Upsilon_{k+p}\, f_{k+p}(z)$ where $\Upsilon_{k+p} \ge 0$ ($k \ge 1$) and $\sum_{k=0}^{\infty} \Upsilon_{k+p} = 1$.

Corollary 6: The extreme points of the class $S^*_{sc,m,q}\, T_p(z)$ are functions $f_{k+p}(z)$ ($k \ge 1, p \in N$) given by Theorem 9.

5. Particular Cases

We have the following interesting relationships with some of the special function classes for suitable choices of parameters, which were investigated recently:

(i) For q = 2, s =1, $a_1 = \lambda + 1$ ($\lambda > -1$) and $a_2 = b_1 = 1$ the class $H_{m,q}(a_1)f(z)$ reduces to the class $D^\lambda f(z)$, where D^λ ($\lambda > -1$) is theRuscheweyh derivative operator (see [10] and [13]).

(ii) For q = 2, s = 1, $a_1 = 2$, $a_2 = 1$ and $b_1 = 2-\mu$ ($\mu \ne 2,3,..$) the Dziok-Srivastava Operator $H_{m,q}(a_1)f(z)$ reduces $\Omega^{\mu}_z f(z)$. Where Ω^{μ}_z is the Srivastava-Owa fractional derivative operator (see [5] and [7]).

(iii) For q = 2, s = 1, a_1 = 2, a_2 = 1 and b_1 = k+1(k > -1) the Dziok-Srivastava Operator $H_{m,q}\,(a_1)f(z)$ reduces to I_k f(z). Where I_k is the Noor integral operator (see [14]).

(iv) For q = 2, s = 1, a_1 = ν+1(ν > -1), a_2 = 1 and b_1 = ν +2 the Dziok-Srivastava Operator $H_{m,q}\,(a_1)f(z)$ reduces to J_ν f(z). Where J_ν is the Generalized Bernardi-Libra-Livingston operator (see [4],[6],[8]).

(v) For q = 2, s = 1, a_1 = a (a > 0), a_2 = 1 and b_1 = c (c >0) the Dziok-Srivastava Operator $H_{m,q}\,(a_1)f(z)$ reduces to L(a,c). Where L(a,c) is the Carlson-Shaffer Operator (see [12]).

(vi) For q = 2, s = 1, a_1 = μ (μ > 0), a_2 = 1 and b_1 = λ+1 (λ > -1) the Dziok-Srivastava Operator $H_{m,q}\,(a_1)f(z)$ reduces $I_{\lambda,\mu}$ f(z). Where $I_{\lambda,\mu}$ is the Choi-Saigo-Srivastava Operator (see [3]).

(vii) For q = 2 and s = 1 the Dziok-Srivastava Operator $H_{m,q}\,(a_1)f(z)$ reduces to the Hohlow Operator $I_{b_1}^{a_1,a_2}$, (see [9]).

REFERENCES

1. Robertson M.S., "on the theory of univalent functions" Ann. Math. , 37(1936), pp374-408...
2. Dziok J., Srivastava H.M., "classes of analytic functions associated with the generalized hyper geometric function", Appl. Math. Comput., 103(1999), pp1-13.
3. Choi J.H., Siago.M. and Srivastava H.M. " Some inclusion properties of a certain family of integral operators", J.Math.Anal.Appl.,276(2002), pp432-445.
4. Bernardi S.D., "convex and starlike univalent functions" Trans. Amer. Math. Soc., 135(1969), pp 429-449.
5. Owa S., "On the distortion theorems". I, Kyungpook Math .J., 18(1978), pp53-59.
6. Libere R.I., "Some classes of regular univalent functions", Proc. Amer. Math. Soc., 16(1969) , pp755-758.
7. Owa S. and Srivastava H.M., "Univalent and starlike generalized hypergeometric functions", Canad. J. Math ., 39(1987), pp 1057-1077.
8. Livingston A.E., "On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc., 17(1966), pp352-357.
9. Hohlow Yu. E., "Operators and operations in the class of univalent functions" Izvestiya Vysshikh Uchebnykh Zavedenii Matematika., 10(1978) (Russian) , pp83-89.
10. Khairnar S.M. and Rajas S.M. , "On a new class of analytic and univalent functions with negative coefficients defined by Ruscheweyh derivative", Int.J.Contemp. Math. Sci., 4(2009), no.34. pp 1653-1664.
11. Nasr M.A. and Aouf M.K. , "Radius of convexity for the class of starlike functions of convex order", Bulletin of the faculty of science. A Physics and Mathematics; (1983), pp153-159.
12. Carlson R.C. and Shaffer D.R., "Starlike and prestralike hypergeometric functions", J. Math. Anal. Appl., 292(2004), pp470-483
13. Ruscheweyh St. , "New Criteria for univalent functions", Proc. Amer. Math. Soc., 49(1975), pp109-115.
14. Noor K.I., "On new classes of integral operators", J. Natur. Geom., 16(1999), pp71-80.
15. Altintaşandö O., özkan, "Starlike, Convex and close to convex functions of complex order", Hacettepe Bulletin of Natural sciences and Engineering. Series B, 28(1991), pp37-4.
16. Aouf M.K., Mostafa A.K., Lashin A.Y. and Munassar B.M., "On a new class of univalent functions with negative coefficients defined by Dziok-Srivastava Operator", J. Math. Anal. Appl., 2(2014), pp117-131.