On Certain Subclasses of Univalent Functions with Negative Coefficients and (m,q)-Starlike with Respect to Certain Points

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ABSTRACT *In this Paper we have introduced and study subclasses of univalent functions with negative coefficient by using Dziok-Srivastava operator defined in punchered open unit disk. Here we have studied coefficient estimates, Distortion Theorems, Extreme Points. These results include many results as particular cases.*

Keywords- *Univalent functions, Coefficient Estimates, Distortion Bounds, Extreme Points.*

INTRODUCTION

Let C denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} \alpha_k z^k \tag{1}$$

which are analytic and univalent in the punchered open unit disk U = {z: $|z| < 1$} and normalized by f' (0) = f (0) +1 = 1. Let f ∈ C given by (1) and g ∈ C given by

$$g(z) = z + \sum_{k=2}^{\infty} \beta_k z^k \quad . \tag{2}$$

We define the convolution product (or Hadamard) of f and g by

$$(f*g)(z) = z + \sum_{k=2}^{\infty} \alpha_k \ \beta_k \ z^k = (g*f)(z); \qquad (z \in U) \tag{3}$$

For +ve real parameters a_1, a_2, -----, a_m and b_1, b_2, ------, b_q.
$\left(b_j \in \mathbb{C}/\mathbb{Z}_0^-, \mathbb{Z}_0^- = 0,-1,-2,\ldots\ldots\ldots; j = 1,2,\ldots\ldots,q\right)$, the generalized hyper geometric function ${}_mF_q$ (a_1, a_2, -----------, a_m, b_1, b_2, -------------, b_q ; z) is defined by

$${}_mF_q\ (a_1, a_2, \ldots\ldots\ldots\ldots\ldots a_m; b_1, b_2, \ldots\ldots\ldots\ldots b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots\ldots\ldots.(a_m)_n}{(b_1)_n \ldots\ldots\ldots.(b_q)_n} \cdot \frac{1}{n!} z^n$$

$(q \leq s+1\,; s, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1,2,\ldots.\};\ z \in U)$ Where$(\ \ )_n$ is the Pochhammer symbol defined in terms of the Gamma function, by

$$(\ \ )_n = \frac{\Gamma(\phi + n)}{\Gamma(\phi)} = \begin{cases} 1 & , n = 0 \\ \phi(\phi+1)\ldots\ldots(\phi+n-1) & , n \in \mathbb{N} \end{cases}$$

For the function

h (a_1, a_2, ---, a_m, b_1, b_2, ----, b_q ; z) =z ${}_qF_s$ (a_1, a_2, ----, a_m, b_1, b_2, ------, b_q ; z),

the Dziok-Srivastava linear operator (see [1]).

$H_{m,q}$ (a_1, a_2, ---------, a_m, b_1, b_2, --------, b_q) : C →C

is defined by the Hadamard product as follows:

$$\begin{aligned} H_{m,q}(a_1, a_2, ----, a_m, b_1, b_2, ----, b_q) f(z) &= h(a_1, a_2, ---, a_m, b_1, b_2, ----, b_q; z)*f(z) \\ &= z + \sum_{n=1}^{\infty} \Gamma_n(a_1)\alpha_n z^n \qquad (z \in U) \end{aligned} \tag{4}$$

Where

$$\Gamma_n(a_1) = \frac{(a_1)_{n-1}\ldots\ldots\ldots(a_m)_{n-1}}{(b_1)_{n-1}\ldots\ldots\ldots(b_q)_{n-1}} \cdot \frac{1}{(n-1)!} \tag{5}$$

For brevity, we write

$H_{m,q}$ (a_1, a_2, ----, a_m, b_1, b_2, ----, b_q ; z) f(z) = $H_{m,q}$ (a_1) f(z)

Definition 1. Let the function f (z) be defined by (1) then $f(z) \in S_{m,q}(z)$ if and only if

$$Re\, \frac{z\left(H_{m,q}(a_1) f(z)\right)'}{H_{m,q}(a_1) f(z)} > 0. \tag{6}$$

Definition 2. Let the function f (z) be defined by (1) then $f(z) \in S_{m,q}^*(z)$ if and only if

$$Re\, \frac{z\left(H_{m,q}(a_1) f(z)\right)'}{H_{m,q}(a_1) f(z) - H_{m,q}(a_1) f(-z)} > 0, \tag{7}$$

these classes of functions are called starlike with respect to symmetric points in U.
Let T denotes the subclasses of C consisting of functions of the form :

$$f(z) = z - \sum_{k=2}^{\infty} \alpha_k z^k \qquad (\alpha_k \ \geq 0.) \tag{8}$$

Definition 3. Let the function f(z) be defined by (8). Then f(z) is said to be m-q starlike with respect to symmetric points if it satisfies the following condition:

$$\left|\frac{z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z)-H_{m,q}(a_1)f(-z)}-1\right|<\beta\left|\frac{\alpha z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z)-H_{m,q}(a_1)f(-z)}+1\right| \quad (9)$$

For some $0<\beta\le 1, 0\le\alpha\le 1$ *and* $z\in U$. We denote the class m-q-starlike with respect to symmetric points by $S^*_{s,m,q}T(z)$.

Definition 4. Let the function f(z) be defined by (8). Then f(z) is said to be m-q starlike with respect to conjugate points if it satisfies the following condition:

$$\left|\frac{z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z)+\overline{H_{m,q}(a_1)f(\bar{z})}}-1\right|<\beta\left|\frac{\alpha z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z)-\overline{H_{m,q}(a_1)f(\bar{z})}}+1\right| \quad (10)$$

For some $0<\beta\le 1, 0\le\alpha\le 1$ *and* $z\in U$. We denote the class m-q-starlike with respect to conjugate points by $S^*_{c,m,q}T(z)$.

Definition 5. Let the function f(z) be defined by (8). Then f(z) is said to be m-q starlike with respect to symmetric conjugate points if it satisfies the following condition:

$$\left|\frac{z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z)-\overline{H_{m,q}(a_1)f(\overline{-z})}}-1\right|<\beta\left|\frac{\alpha z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z)+\overline{H_{m,q}(a_1)f(\overline{-z})}}+1\right| \quad (11)$$

For some $0<\beta\le 1, 0\le\alpha\le 1$ *and* $z\in U$. We denote the class m-q-starlike with respect to symmetric conjugate points by $S^*_{sc,m,q}T(z)$.

2. Coefficient Estimates

Theorem 1. Let the function f be defined by (8) and $H_{m,q}(a_1)f(z)-H_{m,q}(a_1)f(-z)\neq 0$ for z ≠ 0. Then f(z) is in the class $S^*_{s,m,q}T(z)$ if and only if

$$\sum_{k=21}^{\infty}\Gamma_k(a_1)[k(1+\beta\alpha)+(\beta-1)(1-(-1)^k)]\alpha_k<\beta(\alpha+2)-1. \quad (12)$$

Proof: Let $f(z)\in S^*_{c,m,q}T(z)$, then

$$\left|\frac{z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z)-H_{m,q}(a_1)f(-z)}-1\right|<\beta\left|\frac{\alpha z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z)-H_{m,q}(a_1)f(-z)}+1\right|$$

Using (4), that is

$$H_{m,q}(a_1)f(z)=z-\sum_{k=2}^{\infty}\Gamma_k(a_1)\alpha_k z^k$$

and

$$H_{m,q}(a_1)f(-z)=-z-\sum_{k=2}^{\infty}\Gamma_k(a_1)\alpha_k(-1)^k z^k$$

then we have

$$\left|z\left(H_{m,q}(a_1)f(z)\right)'-H_{m,q}(a_1)f(z)+H_{m,q}(a_1)f(-z)\right|$$
$$<\beta\left|\alpha z\left(H_{m,q}(a_1)f(z)\right)'+H_{m,q}(a_1)f(z)-H_{m,q}(a_1)f(-z)\right|$$

that is

$$\left|z-\sum_{k=2}^{\infty}k\Gamma_k(a_1)\alpha_k z^k-z+\sum_{k=2}^{\infty}\Gamma_k(a_1)\alpha_k z^k-z-\sum_{k=2}^{\infty}\Gamma_k(a_1)\alpha_k(-1)^k z^k\right|$$
$$<\beta\left|\alpha z-\sum_{k=2}^{\infty}\alpha k\Gamma_k(a_1)\alpha_k z^k+z-\sum_{k=2}^{\infty}\Gamma_k(a_1)\alpha_k z^k+z+\sum_{k=2}^{\infty}\Gamma_k(a_1)\alpha_k(-1)^k z^k\right|$$

Which readily gives

$$\left|-z-\sum_{k=2}^{\infty}\Gamma_k(a_1)\alpha_k z^k[k-1+(-1)^k]\right|<\beta\left|(\alpha+2)z-\sum_{k=2}^{\infty}\Gamma_k(a_1)\alpha_k z^k[\alpha k+1-(-1)^k]\right|$$

also,

$$|-z| + \sum_{k=2}^{\infty} \Gamma_k(a_1)\alpha_k |z|^k[k-1+(-1)^k] < \beta\,|(\alpha+2)z| - \sum_{k=2}^{\infty} \Gamma_k(a_1)|z|^k\,[\alpha\beta k + \beta - \beta(-1)^k]$$

that is

$$|z| + \sum_{k=2}^{\infty} \Gamma_k(a_1)\alpha_k |z|^k[k-1+(-1)^k] + \sum_{k=2}^{\infty} \Gamma_k(a_1)|z|^k\,[\alpha\beta k + \beta - \beta(-1)^k] < \beta\,(\alpha+2)|z|$$

Which gives

$$\sum_{k=2}^{\infty} \Gamma_k(a_1)[k(1+\beta\alpha) + (\beta-1)(1-(-1)^k)]\alpha_k |z|^k < [\beta\,(\alpha+2)-1]|z|$$

Let $|z| \to 1^-$. So we have

$$\sum_{k=21}^{\infty} \Gamma_k(a_1)[k(1+\beta\alpha) + (\beta-1)(1-(-1)^k)]\alpha_k < \beta\,(\alpha+2)-1.$$

Hence by maximum modulus theore, we have $f(z) \in S^*_{c,m,q}\,T(z)$.

Conversely: - we assume that

$$\left|\frac{\dfrac{z\left(H_{m,q}\,(a_1)f(z)\right)'}{H_{m,q}\,(a_1)f(z) - H_{m,q}\,(a_1)f(-z)} - 1}{\dfrac{\alpha\, z\left(H_{m,q}\,(a_1)f(z)\right)'}{H_{m,q}\,(a_1)f(z) - H_{m,q}\,(a_1)f(-z)} + 1}\right| < \beta$$

that is

$$\left|\frac{-z - \sum_{k=2}^{\infty}\Gamma_k(a_1)\alpha_k\, z^k[k-1+(-1)^k]}{(\alpha+2)\,z - \sum_{k=2}^{\infty}\Gamma_k(a_1)\alpha_k\; z^k\,[\alpha k + 1 - (-1)^k]}\right| < \beta$$

Now since $Re\, f(z) \le |f(z)|$ for all z we have

$$Re\left\{\frac{-z - \sum_{k=2}^{\infty}\Gamma_k(a_1)\alpha_k\, z^k[k-1+(-1)^k]}{(\alpha+2)\,z - \sum_{k=2}^{\infty}\Gamma_k(a_1)\alpha_k\; z^k\,[\alpha k + 1 - (-1)^k]}\right\} < \beta \qquad (13)$$

On the real axis choose values of z, we have $\frac{z\left(H_{m,q}\,(a_1)f(z)\right)'}{H_{m,q}\,(a_1)f(z) - H_{m,q}\,(a_1)f(-z)} - 1$ is real and $H_{m,q}\,(a_1)f(z) - H_{m,q}\,(a_1)f(-z) \neq 0$ for z ≠ 0. Upon clearing denominator in (13) and letting z →1⁻ along the real values, we

$$1 + \sum_{k=2}^{\infty} \Gamma_k(a_1)\alpha_k\,[k-1+(-1)^k] + \sum_{k=2}^{\infty} \Gamma_k(a_1)\,[\alpha\beta k + \beta - \beta(-1)^k] < \beta\,(\alpha+2)$$

This completes the proof of Theorem 1. ▪

Corollary 1: Let the function f (z) defined by (8) be in the class $S^*_{s,m,q}\,T(z)$. Then we have

$$\alpha_k \le \frac{\beta\,(\alpha+2)-1}{\Gamma_k(a_1)[k(1+\beta\alpha) + (\beta-1)(1-(-1)^k)]} \quad (k \ge 2). \qquad (14)$$

The equality in (14) is attained for the function f (z) given by

$$f(z) = z - \frac{\beta\,(\alpha+2)-1}{\Gamma_k(a_1)[k(1+\beta\alpha) + (\beta-1)(1-(-1)^k)]}\, z^k \quad (k \ge 2). \qquad (15)$$

Theorem 2: Let the function f be defined by (8) and $H_{m,q}\,(a_1)f(z) - \overline{H_{m,q}\,(a_1)f(\bar{z})} \neq 0$ for z ≠ 0. Then f (z) is in the class $S^*_{c,m,q}\,T(z)$ if and only if

$$\sum_{k=2}^{\infty} \Gamma_k(a_1)[k(1+\beta\alpha) + 2(\beta-1)]\alpha_k < \beta(\alpha+2)-1. \qquad (16)$$

Corollary 2: Let the function f (z) defined by (8) be in the class $S^*_{c,m,q}\,T(z)$. Then we have

$$\alpha_k \le \frac{\beta(\alpha+2)-1}{\Gamma_k(a_1)[k(1+\beta\alpha) + 2(\beta-1)]} \quad (k \ge 2). \qquad (17)$$

The equality in (17) is attained for the function f (z) given by

$$f(z) = z - \frac{\beta(\alpha+2)-1}{\Gamma_k(a_1)[k(1+\beta\alpha) + 2(\beta-1)]}\; z^k \quad (k \ge 2). \qquad (18)$$

Theorem 3. Let the function f be defined by (8) and $H_{m,q}\,(a_1)f(z) - \overline{H_{m,q}\,(a_1)f(\overline{-z})} \neq 0$ for z ≠ 0. Then f (z) is in the class $S^*_{sc,m,q}\,T(z)$ if and only if

$$\sum_{k=21}^{\infty} \Gamma_k(a_1)[k(1+\beta\alpha) + (\beta-1)(1-(-1)^k)]\alpha_k < \beta\,(\alpha+2)-1. \qquad (19)$$

Corollary 3: Let the function f (z) defined by (8) be in the class $S^*_{sc,m,q}\,T(z)$. Then we have

$$\alpha_k \le \frac{\beta(\alpha+2)-1}{\Gamma_k(a_1)[k(1+\beta\alpha)+(\beta-1)(1-(-1)^k)]} \quad (k \ge 2). \tag{20}$$

The equality in (20) is attained for the function f (z) given by

$$f(z) = z - \frac{\beta(\alpha+2)-1}{\Gamma_k(a_1)[k(1+\beta\alpha)+(\beta-1)(1-(-1)^k)]} z^k \quad (k \ge 2). \tag{21}$$

3. Growth and Distortion Bounds

Theorem 4. Let the function f defined by (8) be in the class $S^*_{s,m,q}\,T(z)$. Then for $|z| = r < 1$, we have

$$|f(z)| \ge r - \frac{\beta(2+\alpha)-1}{2(1+\alpha\beta)\Gamma_2(a_1)} r^2 \tag{22}$$

and

$$|f(z)| \le r + \frac{\beta(2+\alpha)-1}{2(1+\alpha\beta)\Gamma_2(a_1)} r^2 \tag{23}$$

for $z \in U$. The equalities in (22) and (23) are attained for the function given by

$$|f(z)| = z - \frac{\beta(2+\alpha)-1}{2(1+\alpha\beta)\Gamma_2(a_1)} . z^2 \tag{24}$$

at z = r .

Proof:- Since for k ≥ 2

$$2(1+\alpha\beta)\Gamma_2(a_1) \le \Gamma_k(a_1)[k(1+\beta\alpha)+(\beta-1)(1-(-1)^k)]$$

using Theorem 1, we have

$$2(1+\alpha\beta)\Gamma_2(a_1)\sum_{k=2}^{\infty}\alpha_k \le \sum_{k=2}^{\infty}\Gamma_k(a_1)[k(1+\beta\alpha)+(\beta-1)(1-(-1)^k)]\,\alpha_k \le \beta(2+\alpha)-1$$

then

$$\sum_{k=2}^{\infty}\alpha_k \le \frac{\beta(2+\alpha)-1}{2(1+\alpha\beta)\Gamma_2(a_1)} \tag{25}$$

From (25) and (8), we have

$$|f(z)| \ge r - r^2\sum_{k=2}^{\infty}\alpha_k \ge r - \frac{\beta(2+\alpha)-1}{2(1+\alpha\beta)\Gamma_2(a_1)} r^2$$

and

$$|f(z)| \le r + r^2\sum_{k=2}^{\infty}\alpha_k \le r + \frac{\beta(2+\alpha)-1}{2(1+\alpha\beta)\Gamma_2(a_1)} r^2$$

This completes the proof of Theorem 4. ▪

Theorem 5:- Let the function f(z) defined by (8) be in the $S^*_{s,m,q}\,T(z)$. Then for $|z| = r < 1$, we have

$$|f'(z)| \ge 1 - \frac{\beta(2+\alpha)-1}{(1+\alpha\beta)\Gamma_2(a_1)} r \tag{26}$$

and

$$|f'(z)| \le 1 + \frac{\beta(2+\alpha)-1}{(1+\alpha\beta)\Gamma_2(a_1)} r \tag{27}$$

for $z \in U$. The equalities in (22) and (23) are attained for the function f(z) given by (24)

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Proof: - for k ≥ 2. Using theorem 1, we have

$$2(1+\alpha\beta)\Gamma_2(a_1)\sum_{k=2}^{\infty}k\,\alpha_k \le 2\sum_{k=2}^{\infty}\Gamma_k(a_1)[k(1+\beta\alpha)+(\beta-1)(1-(-1)^k)]\,\alpha_k \le 2\{\beta(2+\alpha)-1\}$$

then

$$\sum_{k=2}^{\infty}k\,\alpha_k \le \frac{\beta(2+\alpha)-1}{(1+\alpha\beta)\Gamma_2(a_1)} \tag{28}$$

From (28) & (8), we have

$$|f'(z)| \ge 1 - r\sum_{k=2}^{\infty}k\,\alpha_k \ge 1 - \frac{\beta(2+\alpha)-1}{(1+\alpha\beta)\Gamma_2(a_1)} r$$

And

$$|f^{'(z)}| \le 1 + r\sum_{k=2}^{\infty}k\,\alpha_k \le 1 + \frac{\beta(2+\alpha)-1}{(1+\alpha\beta)\Gamma_2(a_1)} r$$

This completes the proof of Theorem 5. ▪

On similar lines of Theorem 4 and Theorem 5, we can easily prove the following Theorem 6 and Theorem 7 respectively for f (z) belongs to $S^*_{c,m,q}\,T(z)$.

Theorem 6. Let the function f defined by (8) be in the class $S^*_{c,m,q}\,T(z)$. Then for $|z| = r < 1$, we have

$$|f(z)| \ge r - \frac{\beta(\alpha+2)-1}{2\Gamma_2(a_1)\,\beta(\alpha+1)}.r^2 \tag{29}$$

and

$$|f(z)| \le r + \frac{\beta(\alpha+2)-1}{2\Gamma_2(a_1)\,\beta(\alpha+1)}.r^2 \tag{30}$$

for $z \in U$. The equalities in (29) and (30) are attained for the function given by

$$|f(z)| = z - \frac{\beta(\alpha+2)-1}{2\Gamma_2(a_1)\,\beta(\alpha+1)}.z^2 \tag{31}$$

at z = r .

Theorem 7 :- Let the function f(z) defined by (8) be in the $S^*_{c,m,q}\,T(z)$. Then for $|z| = r < 1$, we have

$$|f'(z)| \ge 1 - \frac{\beta(\alpha+2)-1}{\Gamma_2(a_1)\,\beta(\alpha+1)}\,r \tag{32}$$

and

$$|f'(z)| \le 1 + \frac{\beta(\alpha+2)-1}{\Gamma_2(a_1)\,\beta(\alpha+1)}\,r \tag{33}$$

for $z \in U$. The equalities in (32) and (33) are attained for the function f(z) given by (31).

4. Extreme Points

Theorem 8:- Let $f_1(z) = z$ and

$$f_k(z) = z - \frac{\beta\,(\alpha+2)-1}{\Gamma_k(a_1)[k(1+\beta\alpha)+\ (\beta-1)(1-(-1)^k)]}\,z^k \tag{34}$$

where k ≥ 2. Then $f(z) \in S^*_{s,m,q}\,T(z)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \Upsilon_k\, f_k(z) \tag{35}$$

Where $\Upsilon_k \ge 0$ ($k \ge 1$) and $\sum_{k=1}^{\infty} \Upsilon_k = 1$.

Proof: Suppose

$$f(z) = \sum_{k=1}^{\infty} \Upsilon_k\, f_k(z) = z - \sum_{k=2}^{\infty} \frac{\beta\,(\alpha+2)-1}{\Gamma_k(a_1)[k(1+\beta\alpha)+\ (\beta-1)(1-(-1)^k)]}\,\Upsilon_k\, z^k$$

Then we get

$$\sum_{k=2}^{\infty} \frac{\beta\,(\alpha+2)-1}{\Gamma_k(a_1)[k(1+\beta\alpha)+\ (\beta-1)(1-(-1)^k)]}\,.\frac{\Gamma_k(a_1)[k(1+\beta\alpha)+\ (\beta-1)(1-(-1)^k)]}{\beta\,(\alpha+2)-1}\,\Upsilon_k = \sum_{k=2}^{\infty} \Upsilon_k$$
$$= 1 - \Upsilon_1 \le 1.$$

it follows from Theorem 1 that the function $f(z) \in S^*_{s,m,q}\,T(z)$.

Conversely: suppose that $f(z) \in S^*_{s,m,q}\,T(z)$. Again by using Theorem 1, we can show that

$$|\alpha_k| \le \frac{\beta\,(\alpha+2)-1}{\Gamma_k(a_1)[k(1+\beta\alpha)+\ (\beta-1)(1-(-1)^k)]} \quad (k \ge 2). \tag{36}$$

setting

$$|\Upsilon_k| \le \frac{\Gamma_k(a_1)\left[k(1+\beta\alpha)+p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right]}{\beta(\alpha+1-(-1)^p)p+p\,(-1)^p} \tag{37}$$

and

$$\Upsilon_1 = 1 - \sum_{k=2}^{\infty} \Upsilon_k \tag{38}$$

We can see that f(z) can be expressed in the form of (34). This completes the proof of Theorem 8.

Corollary 4: The extreme points of the class $S^*_{s,m,q}\,T(z)$ are functions $f_k(z)$ ($k \ge 2$) given by Theorem 8.

Similar to Theorem 8, we can easily prove the following theorems for $f(z) \in S^*_{c,m,q}\,T(z)$ and $f(z) \in S^*_{sc,m,q}\,T(z)$ classes.

Theorem 9:- Theorem 8:- Let $f_1(z) = z$ and

$$f_k(z) = z - \frac{\beta(\alpha+2)-1}{\Gamma_k(a_1)[k(1+\beta\alpha)+2(\beta-1)]}\,z^k \tag{39}$$

where k ≥ 2. Then $f(z) \in S^*_{s,m,q}\, T(z)$ if and only if it can be expressed in the form $f(z) = \sum_{k=1}^{\infty} \Upsilon_k\, f_k(z)$ where $\Upsilon_k \geq 0$ ($k \geq 1$) and $\sum_{k=1}^{\infty} \Upsilon_k = 1$.

Corollary 5: The extreme points of the class $S^*_{c,m,q}\, T(z)$ are functions $f_k\,(z)$ ($k \geq 2$) given by Theorem 9.

Theorem 10:- Let $f_1(z) = z$ and

$$f_k(z) = z - \frac{\beta\,(\alpha+2)-1}{\Gamma_k(a_1)[k(1+\beta\alpha) + (\beta-1)(1-(-1)^k)]}\, z^k \quad (40)$$

where k ≥ 2. Then $f(z) \in S^*_{sc,m,q}\, T(z)$ if and only if it can be expressed in the form $f(z) = \sum_{k=1}^{\infty} \Upsilon_k\, f_k(z)$ where $\Upsilon_k \geq 0$ ($k \geq 1$) and $\sum_{k=1}^{\infty} \Upsilon_k = 1$.

Corollary 4: The extreme points of the class $S^*_{sc,m,q}\, T(z)$ are functions $f_k\,(z)$ ($k \geq 2$) given by Theorem 10.

5. Particular Cases

We have the following interesting relationships with some of the special function classes for suitable choices of parameters, which were investigated recently:

(viii) For q = 2, s =1, $a_1 = \lambda + 1$ ($\lambda > -1$) and $a_2 = b_1 = 1$ the class $H_{m,q}\,(a_1)f(z)$ reduces to the class $D^\lambda\, f(z), where\ D^\lambda\ (\lambda > -1)$ is theRuscheweyh derivative operator (see [10] and [13]).

(ix) For q = 2, s = 1, $a_1 = 2$, $a_2 = 1$ and $b_1 = 2-\mu$ ($\mu \neq 2,3,..$) the Dziok-Srivastava Operator $H_{m,q}\,(a_1)f(z)$ reduces $\Omega^{\mu}{}_z f(z)$. Where $\Omega^{\mu}{}_z$ is the Srivastava-Owa fractional derivative operator (see [5] and [7]).

(x) For q = 2, s = 1, $a_1 = 2$, $a_2 = 1$ and $b_1 = k+1$($k > -1$) the Dziok-Srivastava Operator $H_{m,q}\,(a_1)f(z)$ reduces to I_k f(z). Where I_k is the Noor integral operator (see [14]).

(xi) For q = 2, s = 1, $a_1 = \nu+1$($\nu > -1$), $a_2 = 1$ and $b_1 = \nu + 2$ the Dziok-Srivastava Operator $H_{m,q}\,(a_1)f(z)$ reduces to J_ν f(z). Where J_ν is the Generalized Bernardi-Libra-Livingston operator (see [4],[6],[8]).

(xii) For q = 2, s = 1, $a_1 = a$ ($a > 0$), $a_2 = 1$ and $b_1 = c$ ($c > 0$) the Dziok-Srivastava Operator $H_{m,q}\,(a_1)f(z)$ reduces to L(a,c). Where L(a,c) is the Carlson-Shaffer Operator (see [12]).

(xiii) For q = 2, s = 1, $a_1 = \mu$ ($\mu > 0$), $a_2 = 1$ and $b_1 = \lambda+1$ ($\lambda > -1$) the Dziok-Srivastava Operator $H_{m,q}\,(a_1)f(z)$ reduces $I_{\lambda,\mu}$ f(z). Where $I_{\lambda,\mu}$ is the Choi-Saigo-Srivastava Operator (see [3]).

(xiv) For q = 2 and s = 1 the Dziok-Srivastava Operator $H_{m,q}\,(a_1)f(z)$ reduces to the Hohlow Operator $I^{a_1,a_2}_{b_1}$, (see [9]).

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