Some new q-integral Inequalities Associated with q-integral Operator

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ABSTRACT *In recent years many authors investigated q-integral inequalities. Therefore q-integral inequalities have become far reaching tools for the development of many branches of pure and applied mathematics. Here we obtain some new fractiona q-integral inequalities associated with q-integral operator.*

Keywords: *Integral inequalities, Riemann-Liouville fractional integral operator, Polya-Szegotype inequalities*

1. Introduction

In recent years the study of fractional q-integral inequalities involving functions of independent variables has been an important research subject in mathematical analysis because the inequality technique is also one of the very useful tools in the study of special functions and theory of approximations.

For our purpose, we begin by recalling the well-known celebrated functional considered by Chebyshev [1] and defined by

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \tag{1}$$

Where g(x) and f(x) ate two integrable functions on [a, b]. If f(x) and g(x) are synchronous on [a, b], i.e.,

$$(f(x)-f(y))(g(x)-g(y)) \geq 0 \tag{2}$$

For any $x, y \in [a,b]$, then T(f, g) ≥ 0. The functional (1) has attracted many researchers attention due to diverse applications in numerical quadrature, transform theory. The function (1) has also been used to yield a number of integral inequalities (see [2], [3], [4], [5],[6], [7], [8]). In 1935, Grss [9] proved the inequalities

$$|T(f,g)| \leq \frac{(K-k)(L-l)}{4}, \tag{3}$$

Where f(x) and g(x) are two bounded functions i.e.,

$$k \leq f(x) \leq K, \qquad l \leq g(x) \leq L \tag{4}$$

For any $k, K, l, L \in \mathbb{R}$ and x, y ∈ [a, b]. Plya and Szeg [10] obtained the following inequalities defined as

$$\frac{\int_a^b f^2(x)dx \int_a^b g^2(x)dx}{\left(\int_a^b f(x)dx \int_a^b g(x)dx\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{KL}{kl}} + \sqrt{\frac{kl}{KL}}\right)^2 \tag{5}$$

Provided f, g satisfy (4) and k,l > 0. Similarly, Dragomir and Diamond proved that [11]

$$|T(f,g)| \leq \frac{(K-k)(L-l)}{4(b-a)^2\sqrt{klKL}}\int_a^b f(x)dx\int_a^b g(x)dx, \tag{6}$$

Where f(x) and g(x) are two positive integrable functions so that

$$0 < k \leq f(x) \leq K < \infty, \qquad 0 < l \leq g(x) \leq L < \infty \tag{7}$$

Recently, Anber and Dahmani [12], by using the Riemann-Liouville fractional integral, presented some interesting integral inequalities of Plya and Szeg type.

For our purpose, we need the following definitions and some properties.

Definition 1: A real valued function f(t) (t > 0) is said to be in the space $C_\mu^n(n, \mu \in \mathbb{R})$ if there exists a real number p > μ such that $f^n(t) = t^p\ \phi(t)$, where $\phi(t) \in C(0, \infty)$.

Here, for the case n=1, we use a simpler notation $C_\mu^1 = C_\mu$.

Definition 2: Let $\mathbb{R}(\alpha) > 0$, and be real or complex numbers. Then a q-analogue of Saigo fractional integral $I_q^{\alpha,\beta,\eta}$ is given by $\left|\frac{\tau}{t}\right| < 1$ by [13]

$$I_q^{\alpha,\beta,\eta}\{f(t)\} := \frac{t^{-\beta-1}}{\Gamma_q(\alpha)} \int_0^t (q\tau/t\,;q)_{\alpha-1} \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m (q^{-\eta};q)_m}{(q^\alpha;q)_m (q;q)_m} . q^{(\eta-\beta)m} (-1)^m\ q^{-\binom{m}{2}}\ (\tau/t-1)_q^m\ f(\tau)\ d_q\tau \quad (8)$$

The integral operator $I_q^{\alpha,\beta,\eta}$ includes both the q-analogue of the Riemann-Liouville and Erdlyl-Kober fractional integral operators given by the following relationships:

$$I_q^{\alpha}\{f(t)\} := I_q^{\alpha,-\alpha,0}\{f(t)\} = \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (q\tau/t\,;q)_{\alpha-1} f(\tau) d_q\tau \qquad (\alpha > 0; 0 < q < 1) \qquad (9)$$

And

$$I_q^{\eta,\alpha}\{f(t)\} := I_q^{\alpha,\eta,0}\{f(t)\} = \frac{t^{-\eta-1}}{\Gamma_q(\alpha)} \int_0^t (q\tau/t\,;q)_{\alpha-1} \tau^\eta f(\tau) d_q\tau \qquad (\alpha > 0; 0 < q < 1) \qquad (10)$$

Where $(a;q)_n$ is the q-shifted factorial defined by

$$(a;q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{k=0}^{n-1}(1 - aq^k) & (n \in \mathbb{N}) \end{cases} \qquad (11)$$

Where $a, q \in \mathbb{C}$ and it is assumed that $a \neq q^{-m}$ $(m \in \mathbb{N}_0)$. The q-shifted factorial for negative subscript is defined by

$$(a;q)_n := \frac{1}{(1-aq^{-1})(1-aq^{-2})\ldots\ldots(1-aq^{-n})} \qquad (n \in \mathbb{N}_0) \qquad (12)$$

We also write

$$(a;q)_\infty\,; = \prod_{k=0}^{\infty}(1 - aq^k) \qquad (a, q \in \mathbb{C}, |q| < 1) \qquad (13)$$

It follows from (11), (12) and (13) that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \qquad (n \in \mathbb{Z}) \qquad (14)$$

Which can be extened to $n = \alpha \in \mathbb{C}$ as follows

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \qquad (\alpha \in \mathbb{C};\ |q| < 1) \qquad (15)$$

Where the principal value of q^α is taken, for f(t) = t^μ in (8), we get the known formula

$$I_q^{\alpha,\beta,\eta}\{f(t)\} = \frac{\Gamma_q(\mu+1)\Gamma_q(\mu-\beta+\eta+1)}{\Gamma_q(\mu-\beta+1)\Gamma_q(\mu+\alpha+\eta+1)} x^{\mu-\beta} \qquad (16)$$

Lemma 1: (Choi and Agarwal [14]) Let 0 < q < 1 and $f : [0, \infty) \to \mathbb{R}$ be a function with $f(t) \geq 0$ for all $t \in [0, \infty)$. Then we have the following inequalities:

(i) Siago fractional q-integral operator of teh function f(t) in (8)

$$I_q^{\alpha,\beta,\eta}\{f(t)\} \geq 0 \qquad (17)$$

For all α > 0 and $\beta, \eta \in \mathbb{R}\ with\ \alpha + \beta > 0\ and\ \eta < 0$.

(ii) The q-analogue of Riemann-Liouville fractional integral operator of the function f(t) of order in (9)

$$I_q^{\alpha}\{f(t)\} \geq 0 \qquad (18)$$

For all α > 0 and $\beta, \eta \in \mathbb{R}\ with\ \alpha + \beta > 0\ and\ \eta < 0$.

(iii) The q-analogue of Erdlyl-Kober fractional integral operator of the function f(t) of order in (10)

$$I_q^{\alpha,\eta}\ \{f(t)\}\ \geq 0 \tag{19}$$

For all α > 0 and $\beta, \eta\ \in\ \mathbb{R}$

2. Certain Fractional q-integral inequalities

Here, we establish Plya-Szag type integral inequalities for functions involving fractional integral operator (8). We begin with lemma involving a q-analogue of Siago fractional integral operator.

Lemma2. Let 0 < q < 1, u and v be two continuous and positive integral functions on $[0, \infty)$ with

$$0 < k_1\ \leq u(\tau) \leq K_1 < \infty, \quad 0 < l_1\ \leq v(\tau) \leq L_1 < \infty\ \ (\tau\ \in [0, t], \in t > 0). \tag{20}$$

Then the following inequality holds true:

$$\frac{\left(I_q^{\alpha,\beta,\eta}\ \{u^2(t)\}\right)\left(I_q^{\alpha,\beta,\eta}\ \{v^2(t)\}\right)}{\left(I_q^{\alpha,\beta,\eta}\ \{u(t)\}\{v(t)\}\right)^2}\ \leq \frac{1}{4}\left(\sqrt{\frac{K_1L_1}{k_1l_1}} + \sqrt{\frac{k_1l_1}{K_1L_1}}\right)^2 \tag{21}$$

For all α > 0 and $\beta, \eta\ \in\ \mathbb{R}\ with\ \alpha + \beta > 0\ and\ \eta < 0.$

Theorem 1: Let 0 < q < 1, f and g be two positive function on $[0, \infty)$ and K, k, L and l be positive real number with inequality (20) holds. Then the following holds true

$$\left|\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}\ t^{-\beta} I_q^{\alpha,\beta,\eta}\ \{f(t)\}\{g(t)\}\right|\ \leq\ \frac{(K-k)(L-l)}{4\sqrt{KLkl}}\ I_q^{\alpha,\beta,\eta}\ \{f(t)\}I_q^{\alpha,\beta,\eta}\ \{g(t)\} \tag{22}$$

For all α > 0 and $\beta, \eta\ \in\ \mathbb{R}\ with\ \alpha + \beta > 0\ and\ \eta < 0.$

Proof: Let f and g be two positive function on $[0, \infty)$. Then for all $\tau, \rho\ \in (0, t)$ with t > 0, we have

$$A(\tau, \rho) =\ (f(\tau) - f(\rho))(g(\tau) - g(\rho)) \tag{23}$$

Or, equivalently

$$A(\tau, \rho) =\ f(\tau)g(\tau) + f(\rho)g(\rho) - f(\rho)g(\tau) - f(\tau)g(\rho) \tag{24}$$

Now, multiplying both side of (24) by

$$\frac{t^{-\beta-1}}{\Gamma_q(\alpha)}(q\tau/t\,;q)_{\alpha-1}\ \sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}.q^{(\eta-\beta)m}(-1)^m\ q^{-\binom{m}{2}}\ (\tau/t-1)_q^m$$

And taking q-integration of the resulting inequality with respect to from 0 to t by using Definition 2, we get

$$\frac{t^{-\beta-1}}{\Gamma_q(\alpha)}\int_0^t(q\tau/t\,;q)_{\alpha-1}\sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}.q^{(\eta-\beta)m}(-1)^m\ q^{-\binom{m}{2}}\ (\tau/t-1)_q^m\ A(\tau,\rho)\ d_q\tau$$

$$= I_q^{\alpha,\beta,\eta}\ \{f(t)\}\{g(t)\} + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}\ t^{-\beta}f(\rho)g(\rho) - g(\rho)I_q^{\alpha,\beta,\eta}\ \{f(t)\} - f(\rho)I_q^{\alpha,\beta,\eta}\ \{g(t)\} \tag{25}$$

Again multiplying both side of above equation by

$$\frac{t^{-\beta-1}}{\Gamma_q(\alpha)}(q\rho/t\,;q)_{\alpha-1}\ \sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}.q^{(\eta-\beta)m}(-1)^m\ q^{-\binom{m}{2}}\ (\rho/t-1)_q^m$$

And taking q-integration of the resulting inequality with respect to from 0 to t and using (8), we get

$$\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t(q\tau/t\,;q)_{\alpha-1}(q\rho/t\,;q)_{\alpha-1}\left\{\sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}q^{(\eta-\beta)m}(-1)^m\ q^{-\binom{m}{2}}\right\}^2\ (\tau/t-1)_q^m\ (\rho/t-1)_q^m\ A(\tau,\rho)\ d_q\tau\, d_q\rho$$

$$= 2\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}\ t^{-\beta}\,I_q^{\alpha,\beta,\eta}\ \{f(t)\}\{g(t)\} - 2I_q^{\alpha,\beta,\eta}\ \{f(t)\}\{g(t)\}$$

(26)

By using Cauchy-Schwaz inequality for double integrals, we have

$$\left|\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t\left(\frac{q\tau}{t};q\right)_{\alpha-1}\left(\frac{q\rho}{t};q\right)_{\alpha-1}\left\{\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}q^{(\eta-\beta)m}(-1)^m\,q^{-\binom{m}{2}}\right\}^2\left(\frac{\tau}{t-1}\right)_q^m\left(\frac{\rho}{t-1}\right)_q^m A(\tau,\rho)\,d_q\tau\,d_q\rho\right|$$

$$\leq\left[\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t\left(\frac{q\tau}{t};q\right)_{\alpha-1}\left(\frac{q\rho}{t};q\right)_{\alpha-1}\left\{\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}q^{(\eta-\beta)m}(-1)^m\,q^{-\binom{m}{2}}\right\}^2\left(\frac{\tau}{t-1}\right)_q^m\left(\frac{\rho}{t-1}\right)_q^m f^2(\tau)\,d_q\tau\,d_q\rho\right.$$

$$+\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t\left(\frac{q\tau}{t};q\right)_{\alpha-1}\left(\frac{q\rho}{t};q\right)_{\alpha-1}\left\{\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}q^{(\eta-\beta)m}(-1)^m\,q^{-\binom{m}{2}}\right\}^2\left(\frac{\tau}{t-1}\right)_q^m\left(\frac{\rho}{t-1}\right)_q^m f^2(\tau)\,d_q\tau\,d_q\rho$$

$$-2\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t\left(\frac{q\tau}{t};q\right)_{\alpha-1}\left(\frac{q\rho}{t};q\right)_{\alpha-1}\left\{\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}q^{(\eta-\beta)m}(-1)^m\,q^{-\binom{m}{2}}\right\}^2\left(\frac{\tau}{t-1}\right)_q^m\left(\frac{\rho}{t-1}\right)_q^m g^2(\tau)\,d_q\tau\,d_q\rho$$

$$+\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t\left(\frac{q\tau}{t};q\right)_{\alpha-1}\left(\frac{q\rho}{t};q\right)_{\alpha-1}\left\{\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}q^{(\eta-\beta)m}(-1)^m\,q^{-\binom{m}{2}}\right\}^2\left(\frac{\tau}{t-1}\right)_q^m\left(\frac{\rho}{t-1}\right)_q^m g^2(\tau)\,d_q\tau\,d_q\rho$$

$$\left.-2\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t\left(\frac{q\tau}{t};q\right)_{\alpha-1}\left(\frac{q\rho}{t};q\right)_{\alpha-1}\left\{\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}q^{(\eta-\beta)m}(-1)^m\,q^{-\binom{m}{2}}\right\}^2\left(\frac{\tau}{t-1}\right)_q^m\left(\frac{\rho}{t-1}\right)_q^m g(\tau)\,g(\rho)\,d_q\tau\,d_q\rho\right]^{\frac{1}{2}}$$

Applying Definition 2, we get

$$\left|\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t\left(\frac{q\tau}{t};q\right)_{\alpha-1}\left(\frac{q\rho}{t};q\right)_{\alpha-1}\left\{\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}q^{(\eta-\beta)m}(-1)^m\,q^{-\binom{m}{2}}\right\}^2\left(\frac{\tau}{t-1}\right)_q^m\left(\frac{\rho}{t-1}\right)_q^m A(\tau,\rho)\,d_q\tau\,d_q\rho\right|$$

$$\leq 2\left[\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}\,t^{-\beta}I_q^{\alpha,\beta,\eta}\{f^2(t)\}-\left(I_q^{\alpha,\beta,\eta}\{f(t)\}\right)^2\right]^{\frac{1}{2}}\cdot\left[\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}\,t^{-\beta}I_q^{\alpha,\beta,\eta}\{g^2(t)\}-\left(I_q^{\alpha,\beta,\eta}\{g(t)\}\right)^2\right]^{\frac{1}{2}}$$

(27)

Applying Lemma 2, we get

$$\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}\,t^{-\beta}I_q^{\alpha,\beta,\eta}\{f^2(t)\}\leq\frac{1}{4}\left(\sqrt{\frac{K}{k}}+\sqrt{\frac{L}{l}}\right)^2\left(I_q^{\alpha,\beta,\eta}\{f(t)\}\right)^2=\left(\frac{(K+k)^2}{4Kk}-1\right)\left(I_q^{\alpha,\beta,\eta}\{f(t)\}\right)^2$$

or

$$\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}\,t^{-\beta}I_q^{\alpha,\beta,\eta}\{f^2(t)\}\leq\frac{(K+k)^2}{4Kk}\left(I_q^{\alpha,\beta,\eta}\{f(t)\}\right)^2 \tag{28}$$

Similarly we get

$$\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}\,t^{-\beta}I_q^{\alpha,\beta,\eta}\{g^2(t)\}\leq\frac{(L+l)^2}{4Ll}\left(I_q^{\alpha,\beta,\eta}\{g(t)\}\right)^2 \tag{29}$$

Finally, by adding (26), (27), (28) and (29), side by side, we arrive at the desired result (22).

In the sequel, we can present another inequality involving the q-fractional integral operator given by (8), asserted by the following lemma.

Lemma 3: Let 0 < q < 1, u and v be two continuous and positive integral functions on $[0,\infty)$ with (20) holds. Then the following inequality holds true:

$$\frac{\left(I_q^{\alpha,\beta,\eta}\{u^2(t)\}\right)\left(I_q^{\gamma,\delta,\zeta}\{v^2(t)\}\right)}{\left(I_q^{\alpha,\beta,\eta}\{u(t)\}\{v(t)\}\right)\left(I_q^{\gamma,\delta,\zeta}\{u(t)\}\{v(t)\}\right)}\leq\frac{1}{4}\left(\sqrt{\frac{K_1L_1}{k_1l_1}}+\sqrt{\frac{k_1l_1}{K_1L_1}}\right)^2 \tag{30}$$

For all α, γ > 0 and $\beta,\eta,\delta,\zeta\in\mathbb{R}$ *with* $\alpha+\beta>0,\gamma+\delta>0$ *and* $\eta,\zeta<0$.

Theorem 2: Let 0 < q < 1, f and g be two positive function on $[0,\infty)$ and K, k, L and l be positive real number with inequality (20) holds. Then the following holds true

$\left|\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}\, t^{-\beta} I_q^{\alpha,\beta,\eta}\{f(t)\}\{g(t)\} + \frac{\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)}\, t^{-\beta} I_q^{\gamma,\delta,\zeta}\{f(t)\}\{g(t)\} -\right.$
$Iq\alpha,\beta,\eta\, ftIq\gamma,\delta,\zeta\, gt$

$$\left. -I_q^{\alpha,\beta,\eta}\{g(t)\} I_q^{\gamma,\delta,\zeta}\{f(t)\}\right| \le \frac{(K-k)(L-l)}{4\sqrt{KLkl}}\, I_q^{\alpha,\beta,\eta}\{f(t)\} I_q^{\gamma,\delta,\zeta}\{g(t)\} \tag{31}$$

For all $\alpha, \gamma > 0$ and $\beta, \eta, \delta, \zeta \in \mathbb{R}$ *with* $\alpha+\beta > 0, \gamma+\delta > 0$ *and* $\eta, \zeta < 0$.

Proof: Let f and g be two positive function on $[0,\infty)$. Then for all $\tau, \rho \in (0,t)$ with $t > 0$, we have

$$A(\tau,\rho) = (f(\tau) - f(\rho))(g(\tau) - g(\rho)) \tag{32}$$

Or, equivalently

$$A(\tau,\rho) = f(\tau)g(\tau) + f(\rho)g(\rho) - f(\rho)g(\tau) - f(\tau)g(\rho) \tag{33}$$

Now, multiplying both side of (24) by

$$\frac{t^{-\beta-1}}{\Gamma_q(\alpha)}(q\tau/t\,;q)_{\alpha-1}\sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}.\,q^{(\eta-\beta)m}(-1)^m\, q^{-\binom{m}{2}}\,(\tau/t-1)_q^m$$

And taking q-integration of the resulting inequality with respect to from 0 to t by using Definition 2, we get

$$\frac{t^{-\beta-1}}{\Gamma_q(\alpha)}\int_0^t(q\tau/t\,;q)_{\alpha-1}\sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}.\,q^{(\eta-\beta)m}(-1)^m\, q^{-\binom{m}{2}}\,(\tau/t-1)_q^m\, A(\tau,\rho)\, d_q\tau$$

$$= I_q^{\alpha,\beta,\eta}\{f(t)\}\{g(t)\} + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}\, t^{-\beta} f(\rho)g(\rho) - g(\rho) I_q^{\alpha,\beta,\eta}\{f(t)\} -$$
$$f(\rho) I_q^{\alpha,\beta,\eta}\{g(t)\} \tag{34}$$

Multiplying both side of above equation by

$$\frac{t^{-\delta-1}}{\Gamma_q(\gamma)}(q\rho/t\,;q)_{\gamma-1}\sum_{n=0}^{\infty}\frac{\left(q^{\gamma+\delta};q\right)_m\left(q^{-\zeta};q\right)_m}{(q^{\gamma};q)_m(q;q)_m}.\,q^{(\zeta-\delta)n}\,(-1)^n\, q^{-\binom{n}{2}}\,(\rho/t-1)_q^n$$

And taking q-integration of the resulting inequality with respect to from 0 to t and using (8), we get

$$\frac{t^{-(\beta+1)}}{\Gamma_q(\alpha)}\frac{t^{-\delta-1}}{\Gamma_q(\gamma)}\int_0^t\int_0^t(q\tau/t\,;q)_{\alpha-1}(q\rho/t\,;q)_{\gamma-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}\frac{\left(q^{\gamma+\delta};q\right)_m\left(q^{-\zeta};q\right)_m}{(q^{\gamma};q)_m(q;q)_m}.$$

$$.\,q^{(\zeta-\delta)n}\,(-1)^n\, q^{-\binom{n}{2}}\,(\rho/t-1)_q^n q^{(\eta-\beta)m}(-1)^m\, q^{-\binom{m}{2}}(\tau/t-1)_q^m\,(\rho/t-1)_q^n\, A(\tau,\rho)\, d_q\tau\, d_q\rho$$

$$= \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}\, t^{-\beta} I_q^{\alpha,\beta,\eta}\{f(t)\}\{g(t)\} + \frac{\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)}\, t^{-\beta} I_q^{\gamma,\delta,\zeta}\{f(t)\}\{g(t)\}$$
$$- I_q^{\alpha,\beta,\eta}\{f(t)\} I_q^{\gamma,\delta,\zeta}\{g(t)\}$$

$$-I_q^{\alpha,\beta,\eta}\{g(t)\} I_q^{\gamma,\delta,\zeta}\{f(t)\}. \tag{32}$$

By using Cauchy-Schwaz inequality for double integrals, by using (32) and applying Definition 2, we get desired result.

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