A Short History of Euler Conjecture and its Disproval

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Received Jan. 03, 2018 Accepted Feb. 05, 2018

ABSTRACT *Euler (1707-1783) has contributed to the concept of Latin Squares among other mathematical concepts. Euler first published the paper regarding Latin Squares with the famous 36 army officers problem (presented to the academy of Sciences in St. Petersburg in 1779, published in 1782.) and he also gave the more complicated concept of orthogonality of Latin Squares. This paper is regarding short history of Euler Conjecture & its disproval.*

Key words: *Latin Squares, Orthogonal Latin Squares, Graeco Latin Squares (GL Sqaures), Euler Conjecture.*

Introduction

Latin Square of order n:

Latin Square of order n is an arrangement of n symbols in a square of n rows and n columns, such that each symbol occurs exactly once in each row and each column.
Note: A Latin Square of any order n ≥ 2 exists.

Orthogonal Latin Squares of order n:

Two Latin Squares of order n are said to be orthogonal if when superimposed every symbol of one square occurs with every symbol of the other square exactly once.
Note: Pair of Orthogonal Latin Squares of order n are also known as Graeco-Latin Square
(GL Square) of order n, since the practice was to use Latin (Roman) letters (A,B,C,…) for one square and to use Greek letters (α, β, γ,…) for the other square.

Example:

	L_1				L_2	
A	B	C		α	β	γ
B	C	A		γ	α	β
C	A	B		β	γ	α

L_1 and L_2 are orthogonal pair of Latin squares of order 3.

The Graeco Latin Square is

(A, α)	(B, β)	(C, γ)
(B, γ)	(C, α)	(A, β)
(C, β)	(A, γ)	(B, α)

If in a set of Latin Squares any two Latin Squares are orthogonal then the set is called Mutually Orthogonal Latin Squares (MOLS) of order n.

Short History of Euler Conjecture

Euler’s 1782 paper was titled *Recherches sur une nouvelle espèce de quarrés magiques* and it contains references to magic squares which shows how Orthogonal Latin Squares can be used to construct magic squares. There is evidence that Euler became interested in Latin Sqaures through his interest in magic squares. These were well known at the time and he worked on them at an early age and returned to them 50 years later. There is brief piece of work on magic squares in his mathematical notebooks in year 1726, later on in 1776 he presented long paper on the topic to the Academy of Sciences of St. Petersburg, both published posthumously.

Neither in 1776 nor in 1782 did Euler use the phrase Orthogonal Latin Square. He never used the term Graeco-Latin Square, for a pair of Orthogonal Latin Squares, which was obviously derived from his work, certainly not the name *Euler Square* or *Eulerian Square*, also name given later. Euler discussed the 4 x

4 square. He was able to find two Orthogonal Latin Squares each with the property that each diagonal also contains all the symbols.

Euler first published regarding the Latin Squares with the paper beginning with the famous 36 army officers problem presented in St. Petersburg in 1779 and published in 1782. Euler classified the officers problem by denoting the regiments by Latin letters a, b, c, d, e, f and the ranks by Greek letters α, β, γ, δ, ε, ζ. He explained that the task to arrange 36 squares of a Latin and Greek letters in 6 x 6 array, so that each row and column contains each Latin and Greek letter just once. He stated that he could not to do so and generalized that GL Square of order 4n+2, where n is non-negative integer, is not possible. This is known as Euler's conjecture. In other words, GL Square of order 2, 6, 10, 14 … are not possible. This conjecture remained as conjecture for several years.

Disproval of Conjecture

In year 1900, scientist Tarry, confirmed by trial and error method that this conjecture is true for order 6. But in 1959, R.C. Bose and S.S. Shirkhande, working in USA, proved that conjecture is not true for order 4n+2, for integer $n \geq 2$. They also proved theoretically that the conjecture is true for $n = 1$. So, Graeco-Latin Square of order 6 is still not possible but it is possible to construct Graeco-Latin Square of order 10, 14 … This findings appeared as news on 26th April, 1959 in the reputed daily New York Times. So Bose and Shirkhande were known as Euler Crackers. Later on, Bose, Shirkhande and Parker constructed GL Square of many large orders. Meantime, Keshav Menon in India gave a method of construction of GL Square of order 3m+1, where GL Square of order m exists. This method appeared in 1961 in reputed journal *Sankhya* published by Indian Statistical Institute, Calcutta. So we can construct GL Square of order 10, 22 since GL Square of order m = 3, 7 exists. It can be seen that 10, 22 are values of 4n+2 for n = 2, 5. In fact, it was known for last several years how to construct GL Square of order s, where s is prime or prime power.

Conclusions and Future Study

GL Squares are well-known in Design of Experiments, a significant field in subject of Statistics. Generalization of the concept of Latin Squares and orthogonality of Latin Squares is done. The condition that every element appears once in each row and each column in a Latin Square is replaced by the condtion that it appears the same number of times in each row and each column. Such squares are called Frequency Squares or F-Sqaures. The usefulness of introducing and investigating properties of F-Squares could be justified in two directions. F-Squares have meaningful application in laying out experimental design. Their properties prove useful in the studies of existence of Orthogonal Latin Squares and other combinatorial problems.

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