REPRESENTATIONS BY THE FORM $f(x_1, x_2, x_3)$ WITH MUTUALLY SIMPLE ODDINVARIANTS $[d, k]$

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ABSTRACT *This paper deals with the positive form $f(x_1, x_2, x_3)$ over the integers with mutually simple odd invariants $[d, k]$. In this paper we consider a prime number q which does not divide $2k$ and a positive number m prime to q such that we have a primitive congruence $f(x_1, x_2, x_3) \equiv m(mod\ 2^3d^2km)$. Then it is proved that the number of representations by form lies between the number of classes of integral primitive binary quadratic forms.*

Keywords: *Quadratic Forms, Primitive, Representations, Congruence and Mutually Odd Invariants.*

INTRODUCTION

In the paper [4] it is proved that the number of representations given by the form is greater that the number of classes of binary forms. A primitive quadratic form over the field of integers with odd invariants is considered and another form mutually primitive to it.The present paper considers a prime number q which does not divide $2k$ and a positive number m prime to q such that we have a primitive congruence $f(x_1, x_2, x_3) \equiv m(mod\ 2^3d^2km)$.

REPRESENTATION OF INTEGERS BY THEPOSITIVEFORM $f(x_1, x_2, x_3)$ OVER THE INTEGRES WITH MUTUALLY SIMPLE ODDINVARIANTS $[d, k]$

Now the following theorem gives on the representations by positive quadratic form in three variables with mutually odd simple prime invariants.

Theorem: Let $f = f(x_1, x_2, x_3)$ be a positive integral primitive quadratic form in three variables with odd mutually simple invariant $[d, k]$. Let q is a prime number which does not divide $2k$. Let h is a positive integer relatively prime to $2dk$ and b_1, b_2, b_3 are the integers satisfying the condition that $\gcd(h, b_1, b_2, b_3) = 1$. Consider a positive number m prime to q such that we have a primitive congruence

$$f(x_1, x_2, x_3) \equiv m(mod\ 2^3d^2km) \qquad (1)$$

as well as the conditions

$$f(b_1, b_2, b_3) \equiv m(mod\ h) \qquad (2)$$

$$\left(\frac{-km}{q}\right) = 1 \qquad (3)$$

Let $\Delta_{f,m}$ is the field for the form $f(x_1, x_2, x_3) = m$ with the condition that $\gamma > 0$. We denote $r_{h,b_1,b_2,b_3}(\Delta_{f,m})$ as the number of primitive representations of integers (x_1, x_2, x_3) of number m by form f for which the following condition hold

$$(x_1, x_2, x_3) \equiv (b_1, b_2, b_3)(mod\ h) \qquad (4)$$

Then there exist constants $m_0, x > 0$ and $x' > 0$ depending only on k, d, q, h, γ, u in the field $\Delta_{f,m}$ such that when $m \geq m_0$, we have

$$xg(-km) < r_{h,b_1,b_2,b_3}(\Delta_{f,m}) < x'g(-km) \qquad (5)$$

where $g(-km)$ are the number of classes of primitive integer positive quadratic forms in two variables with determinant km.

Proof: The upper bound of the condition (5) is trivial (Chetna and Singh [2]). Therefore, we prove a lower boundby the following three cases:

1) q is relatively prime to d_F.
2) $q \setminus h$ therefore q is prime to $2dk$.

3) $q \setminus d$.

Let q prime to $2dk$.By (3) there exist an integer l satisfying congruence

$l^2 + km \equiv 0(mod\ q)$

We choose a primitive form$A_1(mod\ q)$ with the condition$(A_1) \equiv 0(mod\ q)$. We also suppose that

$$r_1 = q, A_2 = 1, r_2 = 1, r = r_1 r_2 = q, L_0 = b_1 i_1 + b_2 i_2 + b_2 i_3$$

Each integral primitive form $L = x_1 i_1 + x_2 i_2 + x_2 i_3$with norms km has one-to-one correspondence primitive form(x_1, x_2, x_3) of number m of form $f(x_1, x_2, x_3) = m$. Therefore, by (Chetna [4]), we have

$r_{h,b_1,b_2,b_3}(\Delta_{f,m}) > r_{h,L_0}(\Delta_{f,m}, r_1, A_1, r_2, A_2, l) > xg(-km)$

So in this case we obtain the required estimate (5).

Now we prove the second case. Let q divides $h$thus wehave$h = q^{z'} h_1$, where $z' \geq 1$ and the numbers $q, h_1$and $2dk$are relatively primepair-wise. By (Oh [6]) there exists an integer l, satisfying the congruence

$$l^2 + km \equiv 0(mod\ q^{2z'}) \qquad (6)$$

Let $L_0 = b_1 i_1 + b_2 i_2 + b_2 i_3$ and suppose that

$r_1 = r_2 = q^{z'}, A_1 = A_2 = l - L_0, r = r_1 r_2 = q^{2z'}$

Then condition (2) gives us

$N(A_1) = N(A_2) = N(l - L_0) = N(l + L_0) =$

$$= l^2 + kf(b_1, b_2, b_3) \equiv l^2 + km \equiv 0(mod\ q^{z'}) \qquad (7)$$

where form$A_2 A_1$ is primitive $(mod\ q)$, thus by (7) and (6),we have

$A_2 A_1 \equiv 2l(l + L_0)(mod\ q)$

where$l$ is an integer prime to q and we assume that$\gcd(q, km) = 1$.

By (Chetna and Singh [3]),there $> xg(-km)$ primitive integer form$L$with norms km with conditions

$$L \equiv L_0(mod\ h_1), (l + L)A_1 \equiv A_2(l + L) \equiv 0\left(mod\ q^{z'}\right) \qquad (8)$$

Using the congruence (8) and (7), we obtain

$(L - L_0)A_1 = (l + L)A_1 - (l + L_0)A_1 =$

$= (l + L)A_1 - N(l + L_0) \equiv 0(mod\ q^{z'})$

$A_2((L - L_0) = A_2(l + L) - N(l + L_0) \equiv 0(mod\ q^{z'})$

Therefore by (Burton [1]) we have

$$L - L_0 \equiv 0, L \equiv L_0\left(mod\ q^{z'}\right) \qquad (9)$$

Congruence (8) and (9) with the co-primes$q^{z'}$ and h_1 leads us to congruence

$L \equiv L_0(mod\ h)$

Hence, we get the number of representations $m > xg(-km)$by the form$f$ where the congruence (4) and the inequalities (5) are valid.

Finally, we consider the third case. Let us suppose that$q$ divides d, then we consider $d = q^{z'} d_1$, where $q, d_1$and h are pair-wise co-prime. Since there exists a positive integral primitive quadratic form in three variables with invariant $[d_1, k]$, then by (Shimura [7])

$$f(x_1,x_2,x_3) = \theta\left(\sum_{i=1}^{3} c_{1i}x_i, \sum_{i=1}^{3} c_{2i}x_i, \sum_{i=1}^{3} c_{3i}x_i\right)(10)$$

where(c_{ij}) is a 3×3 matrix of integers with determinant $q^{z'}$ is obtainedby linear integer substitution

$$\begin{cases} \overline{x_1} = c_{11}x_1 + c_{12}x_2 + c_{13}x_3 \\ \overline{x_2} = c_{21}x_1 + c_{22}x_2 + c_{23}x_3 \\ \overline{x_1} = c_{31}x_1 + c_{32}x_2 + c_{33}x_3 \end{cases} \qquad (11)$$

Determinant $q^{z'}$ has a one-to-one correspondence between the points of the form$f(x_1,x_2,x_3) = m$ and the points of the form$\theta(\overline{x_1},\overline{x_2},\overline{x_3}) = m$ with the same form and with the same condition$\gamma > 0$. By (1) and (2) we can choose the integersb'_1, b'_2, b'_3 such that

$$f(b'_1, b'_2, b'_3) \equiv m\left(mod\ hq^{z'}\right), (b'_1, b'_2, b'_3) \equiv (b_1, b_2, b_3)(mod\ h) \quad (12)$$

Consider the integers b'_1, b'_2, b'_3 in the equations

$$b'_i = \sum_{i=}^{3} c_{ij}\bar{b}_j (i = 1,2,3) \qquad (13)$$

Thenby (Ben [5]), we have $\theta(b'_1, b'_2, b'_3) \equiv m(mod\ hq^{z'})$

And by comparing we can say that

$$\theta(\overline{x_1},\overline{x_2},\overline{x_3}) \equiv m(mod\ 2^3 d_1^2 km)$$

has a primitive solution. Therefore, according to the preceding result the primitive representations of the number $m > xg(-km)$by form θ with the additional condition

$$(\overline{x_1},\overline{x_2},\overline{x_3}) \equiv (b'_1, b'_2, b'_3)\left(mod\ hq^{z'}\right) \qquad (14)$$

Let $(\overline{x_1},\overline{x_2},\overline{x_3})$be one of the point. Then the point $(\overline{x_1},\overline{x_2},\overline{x_3})$ according to (11) corresponds to the point (x_1,x_2,x_3), thus we have

$$x_i = \frac{1}{q^{z'}} \sum_{j=1}^{3} \overline{c_{ij}}\,\overline{x}_j (i = 1, 2, 3) \qquad (15)$$

where$\overline{c_{ij}}$ is the cofactor of the element c_{ij}in the matrix (c_{ij}). On comparing the equalities (15), (13) and (14) give us

$$x_i = \frac{1}{q^{z'}} \sum_{j=1}^{3} \overline{c_{ij}}\,\overline{x}_j \equiv \frac{1}{q^{z'}} \sum_{j=1}^{3} \overline{c_{ij}}\, b'_j = \frac{1}{q^{z'}} \sum_{j=1}^{3} \overline{c_{ij}} \sum_{\lambda=1}^{3} c_{j\lambda}\, \overline{b_\lambda} = \bar{b}_i \equiv b_i(mod\ h)$$

Therefore (x_1,x_2,x_3) is an integral representation of a primitive $m$by the form f satisfying (4). For different $(\overline{x_1},\overline{x_2},\overline{x_3})$ we have different(x_1,x_2,x_3). Thus, we have proved (5) in the last case. Hence, the theorem is proved.

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