

# REPRESENTATIONS BY THE FORM $f(x_1, x_2, x_3)$ WITH MUTUALLY SIMPLE ODD INVARIANTS $[d, k]$

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## ABSTRACT

This paper deals with the positive form  $f(x_1, x_2, x_3)$  over the integers with mutually simple odd invariants  $[d, k]$ . In this paper we consider a prime number  $q$  which does not divide  $2k$  and a positive number  $m$  prime to  $q$  such that we have a primitive congruence  $f(x_1, x_2, x_3) \equiv m \pmod{2^3 d^2 km}$ . Then it is proved that the number of representations by form lies between the number of classes of integral primitive binary quadratic forms.

**Keywords:** Quadratic Forms, Primitive, Representations, Congruence and Mutually Odd Invariants.

## INTRODUCTION

In the paper [4] it is proved that the number of representations given by the form is greater than the number of classes of binary forms. A primitive quadratic form over the field of integers with odd invariants is considered and another form mutually primitive to it. The present paper considers a prime number  $q$  which does not divide  $2k$  and a positive number  $m$  prime to  $q$  such that we have a primitive congruence  $f(x_1, x_2, x_3) \equiv m \pmod{2^3 d^2 km}$ .

## REPRESENTATION OF INTEGERS BY THE POSITIVE FORM $f(x_1, x_2, x_3)$ OVER THE INTEGERS WITH MUTUALLY SIMPLE ODD INVARIANTS $[d, k]$

Now the following theorem gives on the representations by positive quadratic form in three variables with mutually odd simple prime invariants.

**Theorem:** Let  $f = f(x_1, x_2, x_3)$  be a positive integral primitive quadratic form in three variables with odd mutually simple invariant  $[d, k]$ . Let  $q$  is a prime number which does not divide  $2k$ . Let  $h$  is a positive integer relatively prime to  $2dk$  and  $b_1, b_2, b_3$  are the integers satisfying the condition that  $\gcd(h, b_1, b_2, b_3) = 1$ .

Consider a positive number  $m$  prime to  $q$  such that we have a primitive congruence

$$f(x_1, x_2, x_3) \equiv m \pmod{2^3 d^2 km} \quad (1)$$

as well as the conditions

$$f(b_1, b_2, b_3) \equiv m \pmod{h} \quad (2)$$

$$\left(\frac{-km}{q}\right) = 1 \quad (3)$$

Let  $\Delta_{f,m}$  is the field for the form  $f(x_1, x_2, x_3) = m$  with the condition that  $\gamma > 0$ . We denote  $r_{h, b_1, b_2, b_3}(\Delta_{f,m})$  as the number of primitive representations of integers  $(x_1, x_2, x_3)$  of number  $m$  by form  $f$  for which the following condition hold

$$(x_1, x_2, x_3) \equiv (b_1, b_2, b_3) \pmod{h} \quad (4)$$

Then there exist constants  $m_0, x > 0$  and  $x' > 0$  depending only on  $k, d, q, h, \gamma, u$  in the field  $\Delta_{f,m}$  such that when  $m \geq m_0$ , we have

$$xg(-km) < r_{h, b_1, b_2, b_3}(\Delta_{f,m}) < x'g(-km) \quad (5)$$

where  $g(-km)$  are the number of classes of primitive integer positive quadratic forms in two variables with determinant  $km$ .

**Proof:** The upper bound of the condition (5) is trivial (Chetna and Singh [2]). Therefore, we prove a lower bound by the following three cases:

- 1)  $q$  is relatively prime to  $d$ .
- 2)  $q \nmid h$  therefore  $q$  is prime to  $2dk$ .

3)  $q \nmid d$ .

Let  $q$  prime to  $2dk$ . By (3) there exist an integer  $l$  satisfying congruence

$$l^2 + km \equiv 0 \pmod{q}$$

We choose a primitive form  $A_1 \pmod{q}$  with the condition  $(A_1) \equiv 0 \pmod{q}$ . We also suppose that

$$r_1 = q, A_2 = 1, r_2 = 1, r = r_1 r_2 = q, L_0 = b_1 i_1 + b_2 i_2 + b_2 i_3$$

Each integral primitive form  $L = x_1 i_1 + x_2 i_2 + x_2 i_3$  with norms  $km$  has one-to-one correspondence primitive form  $(x_1, x_2, x_2)$  of number  $m$  of form  $f(x_1, x_2, x_2) = m$ . Therefore, by (Chetna [4]), we have

$$n_{h, b_1, b_2, b_3}(\Delta_{f, m}) > n_{h, L_0}(\Delta_{f, m}, r_1, A_1, r_2, A_2, l) > xg(-km)$$

So in this case we obtain the required estimate (5).

Now we prove the second case. Let  $q$  divides  $h$  thus we have  $h = q^{z'} h_1$ , where  $z' \geq 1$  and the numbers  $q, h_1$  and  $2dk$  are relatively prime pair-wise. By (Oh [6]) there exists an integer  $l$ , satisfying the congruence

$$l^2 + km \equiv 0 \pmod{q^{2z'}} \quad (6)$$

Let  $L_0 = b_1 i_1 + b_2 i_2 + b_2 i_3$  and suppose that

$$r_1 = r_2 = q^{z'}, A_1 = A_2 = l - L_0, r = r_1 r_2 = q^{2z'}$$

Then condition (2) gives us

$$\begin{aligned} N(A_1) &= N(A_2) = N(l - L_0) = N(l + L_0) = \\ &= l^2 + kf(b_1, b_2, b_3) \equiv l^2 + km \equiv 0 \pmod{q^{2z'}} \end{aligned} \quad (7)$$

where form  $A_2 A_1$  is primitive  $\pmod{q}$ , thus by (7) and (6), we have

$$A_2 A_1 \equiv 2l(l + L_0) \pmod{q}$$

where  $l$  is an integer prime to  $q$  and we assume that  $\gcd(q, km) = 1$ .

By (Chetna and Singh [3]), there  $> xg(-km)$  primitive integer form  $L$  with norms  $km$  with conditions

$$L \equiv L_0 \pmod{h_1}, (l + L)A_1 \equiv A_2(l + L) \equiv 0 \pmod{q^{z'}} \quad (8)$$

Using the congruence (8) and (7), we obtain

$$\begin{aligned} (L - L_0)A_1 &= (l + L)A_1 - (l + L_0)A_1 = \\ &= (l + L)A_1 - N(l + L_0) \equiv 0 \pmod{q^{z'}} \\ A_2((L - L_0) &= A_2(l + L) - N(l + L_0) \equiv 0 \pmod{q^{z'}} \end{aligned}$$

Therefore by (Burton [1]) we have

$$L - L_0 \equiv 0, L \equiv L_0 \pmod{q^{z'}} \quad (9)$$

Congruence (8) and (9) with the co-primes  $q^{z'}$  and  $h_1$  leads us to congruence  $L \equiv L_0 \pmod{h}$

Hence, we get the number of representations  $m > xg(-km)$  by the form  $f$  where the congruence (4) and the inequalities (5) are valid.

Finally, we consider the third case. Let us suppose that  $q$  divides  $d$ , then we consider  $d = q^{z'} d_1$ , where  $q, d_1$  and  $h$  are pair-wise co-prime. Since there exists a positive integral primitive quadratic form in three variables with invariant  $[d_1, k]$ , then by (Shimura [7])

$$f(x_1, x_2, x_3) = \theta \left( \sum_{i=1}^3 c_{1i}x_i, \sum_{i=1}^3 c_{2i}x_i, \sum_{i=1}^3 c_{3i}x_i \right) \quad (10)$$

where  $(c_{ij})$  is a  $3 \times 3$  matrix of integers with determinant  $q^2$  is obtained by linear integer substitution

$$\begin{cases} \bar{x}_1 = c_{11}x_1 + c_{12}x_2 + c_{13}x_3 \\ \bar{x}_2 = c_{21}x_1 + c_{22}x_2 + c_{23}x_3 \\ \bar{x}_3 = c_{31}x_1 + c_{32}x_2 + c_{33}x_3 \end{cases} \quad (11)$$

Determinant  $q^2$  has a one-to-one correspondence between the points of the form  $f(x_1, x_2, x_3) = m$  and the points of the form  $\theta(\bar{x}_1, \bar{x}_2, \bar{x}_3) = m$  with the same form and with the same condition  $\gamma > 0$ . By (1) and (2) we can choose the integers  $b'_1, b'_2, b'_3$  such that

$$f(b'_1, b'_2, b'_3) \equiv m \pmod{hq^2}, (b'_1, b'_2, b'_3) \equiv (b_1, b_2, b_3) \pmod{h} \quad (12)$$

Consider the integers  $b'_1, b'_2, b'_3$  in the equations

$$b'_i = \sum_{j=1}^3 c_{ij} \bar{b}_j \quad (i = 1, 2, 3) \quad (13)$$

Then by (Ben [5]), we have  $\theta(b'_1, b'_2, b'_3) \equiv m \pmod{hq^2}$

And by comparing we can say that

$$\theta(\bar{x}_1, \bar{x}_2, \bar{x}_3) \equiv m \pmod{2^3 d_1^2 km}$$

has a primitive solution. Therefore, according to the preceding result the primitive representations of the number  $m > xg(-km)$  by form  $\theta$  with the additional condition

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3) \equiv (b'_1, b'_2, b'_3) \pmod{hq^2} \quad (14)$$

Let  $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$  be one of the point. Then the point  $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$  according to (11) corresponds to the point  $(x_1, x_2, x_3)$ , thus we have

$$x_i = \frac{1}{q^2} \sum_{j=1}^3 \bar{c}_{ij} \bar{x}_j \quad (i = 1, 2, 3) \quad (15)$$

where  $\bar{c}_{ij}$  is the cofactor of the element  $c_{ij}$  in the matrix  $(c_{ij})$ . On comparing the equalities (15), (13) and (14) give us

$$x_i = \frac{1}{q^2} \sum_{j=1}^3 \bar{c}_{ij} \bar{x}_j \equiv \frac{1}{q^2} \sum_{j=1}^3 \bar{c}_{ij} b'_j = \frac{1}{q^2} \sum_{j=1}^3 \bar{c}_{ij} \sum_{\lambda=1}^3 c_{j\lambda} \bar{b}_\lambda = \bar{b}_i \equiv b_i \pmod{h}$$

Therefore  $(x_1, x_2, x_3)$  is an integral representation of a primitive  $m$  by the form  $f$  satisfying (4). For different  $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$  we have different  $(x_1, x_2, x_3)$ . Thus, we have proved (5) in the last case. Hence, the theorem is proved.

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