A NOTE ON UNIT GRAPH OF A RING

SATYANARAYANA BHAVANARI 1 & SRINIVASULU DEVANABOINA2 & MALLIKARJUNA BHAVANARI3

1Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar – 522 510, Andhra Pradesh, INDIA.
2Department of BSH, NRI Institute of Technology, Agiripalli- 521 212, Andhra Pradesh, INDIA.
3Institute of Energy Engineering, Department of Mechanical Engineering, National Central University Jhongli, Taoyuan, TAIWAN – 32001, R.O.C.

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ABSTRACT: The zero divisor graph of a commutative ring R was introduced by Beck [1], and later studied by Vasanthakandasamy [24]. In this short note, we introduced the concept namely “Unit graph of type-1” denoted by UG1(R) in Associative rings R and announced few important fundamental results. We proved that UG1(Zn) contains at least one vertex of degree 1 and if F is a field with |F| is odd, then UG1(F) contains at least one vertex of degree 1. We included some examples.

Key Words: Graph, Unit graph of a ring, Star graph

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1.1 Introduction
Let G = (V, E) be a graph consist of a finite non-empty set V of vertices and finite set E of edges such that each edge e_k is identified as an unordered pair of vertices {v_i, v_j}, where V_i, V_j are called end points of e_k. The edge e_k is also denoted by either v_iV_j or v_jV_i. We also write G(V, E) for the graph. Vertex set and edge set of G are also denoted by V(G) and E(G) respectively. An edge associated with a vertex pair {v_i, v_j} is called a self-loop. The number of edges associated with the vertex is the degree of the vertex, and d(v) denotes the degree of the vertex v. If there is more than one edge associated with a given pair of vertices, then these edges are called parallel edges or multiple edges. A graph that does not have self-loops or parallel edges is called a simple graph. We consider simple graphs only.

1.2 Definitions: (i) A graph G(V, E) is said to be a star graph if there exists a fixed vertex v (called the center of the star graph) such that E = {vu / u ∈ V and u ≠ v}. A star graph is said to be an n-star graph if the number of vertices of the graph is n.
(ii) In a graph G, a subset S of V(G) is said to be a dominating set if every vertex not in S has a neighbour in S. The domination number, denoted by γ(G) is defined as min{|S| / S is a dominating set in G}.
(iii) (Vasanthakandasamy and FlorentinSmarandache [24]) A graph G = (V’, E’) is said to be the zero divisor graph of a commutative ring R if V = R and
E = \{xy / x ≠ y, x, y ∈ R, x ≠ 0 ≠ y, xy = 0\} ∪ \{x0 / 0 ≠ x ∈ R\} where xy denotes an edge between x, y ∈ V.
This definition ‘zero divisor graph’ is same as that of Beck [1988].

1.3 Notation: (i) We denote zero divisor graph of ring R by ZDG(R)
(ii) In the graph ZDG(R), we have that V(ZDG(R)) = R and
E(ZDG(R)) = \{xy / x ≠ y, x, y ∈ R, x ≠ 0 ≠ y, xy = 0\} ∪ \{x0 / 0 ≠ x ∈ R\}

1.4 Examples(i): (Vasanthakandasamy and FlorentinSmarandache [24]) Consider ZDG(R) with R = Z_n. We know that R = Z_n = {0, 1, 2, 3, 4, 5, 6, 7}. SoV(ZDG(R)) = {0, 1, 2, 3, 4, 5, 6, 7}. Since 2.4 = 4.6 = 0 (mod 8), there exist edges between the vertices 2 and 4, 4 and 6. Since ‘0’ is adjacent to all the elements in R, we get 01, 02, 03, 04, 05, 06, 07 ∈ E(ZDG(R)).

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2.1 DEFINITION: Let R be an associative ring with 1. Write U(R) the set of all units in R. A graph G = (V, E) is said to be the unit graph of type-1 of R if \( V = \{1 \neq x, x \in U(R)\} \cup \{xy, x, y \in U(R), x \neq 1 \neq y \text{ and } xy = 1\} \).

If R is a commutative ring, then this definition of ‘Unit graph of type-1’ coincide with the definition of ‘Unit graph’ mentioned by Vasanthakand/aswamy (2009).

2.2 Notation: (i) The unit graph of type-1 of a ring R is denoted by UG1(R).

(ii) In the graph UG1(R) we have that \( V(UG1(R)) = U(R) \) and \( E(UG1) = \{1x / 1 \neq x, x \in U(R)\} \cup \{xy, x, y \in U(R), x \neq 1 \neq y \text{ and } xy = 1\} \).

2.3 Example: Let us construct the graph UG1(R), where \( R = \mathbb{Z}_9, U(R) = \{1, 3, 5, 7\} \). So \( V(UG1(R)) = \{1, 3, 5, 7\} \). Since ‘1’ is adjacent to all units in R, we have \( \{13, 15, 17\} \). There are no other edges because \( \{xy, x, y \in U(R), x \neq 1 \neq y \text{ and } xy = 1\} = \emptyset \). So \( E(UG1) = \{13, 15, 17\} \) and the graph UG1(R) is given in the Fig. 2.3.
2.4 Example: Let us construct the graph \( UG1(R) \), where \( R = \mathbb{Z}_9 \). Here \( U(R) = \{1, 2, 4, 5, 7, 8\} \) and \( V(UG1(R)) = \{1, 2, 4, 5, 7, 8\} \). Since ‘1’ is adjacent to all units in \( R \), we have the edges \( 12, 14, 15, 17, 18, 25, 47 \). Now \( UG1(R) \) is given in Fig. 2.4.

2.5 Result: (i) For any ring \( R \), both the graphs \( ZDG(R) \) and \( UG1(R) \) are simple graphs.

(ii) There are no common edges in \( ZDG(R) \) and \( UG1(R) \).

Proof: (i) is clear.

(ii) We have to verify that \( E(ZDG(R)) \cap E(UG1(R)) = \emptyset \).

In a contrary way, suppose that \( xy \in E(ZDG(R)) \cap E(UG1(R)) \). Now \( xy \in E(UG1(R)), x, y \in U(R) \), \( x \neq 0 \) and \( y \neq 0, \) since \( y \in U(R) \) there exists \( 0 \neq z \in R \) such that \( yz = 1 \).

Since \( xy \in E(ZDG(R)) \), we have that \( xy = 0, x \neq 0, y \neq 0 \). Now \( 0 = 0, z = (xy)z = x(yz) = x.1 = x, \) a contradiction.

Hence \( E(UDG(R)) \cap E(UG1(R)) = \emptyset \).

The proof is complete.

2.6 Note: Examples 1.4(i), 1.4(ii), 2.3 & 2.4 provides two illustrations for \( E(ZDG(R)) \cap E(UG1(R)) = \emptyset \).

2.7 Result: \( UG1(\mathbb{Z}_n) \) contains at least one vertex of degree 1.

Proof: Consider \( \mathbb{Z}_n \).

Since \( (n - 1)^2 = n^2 - 2n + 1 \equiv 1(\text{mod } n) \), we have that \( n - 1 \) is a self invertible element. By definition of \( UG1(R) \), the vertex \( n - 1 \) and the vertex 1 are connected by an edge; and there is no other vertex adjacent to \( n - 1 \). Hence the degree of the vertex \( n - 1 \) is 1.

2.8 Result: If \( F \) is a field with \( |F| \) odd, then \( UG1(F) \) contains at least one vertex of degree 1.

Proof: Suppose \( |F| \) odd. Then \( F \setminus \{0\} \) is a multiplicative group with \( |F \setminus \{0\}| \) even. Since all the elements of \( F \setminus \{0\} \) are units we get that \( V(UG1(F)) = F \setminus \{0\} \).

Also \( 1 \in F \setminus \{0\} \) and \( 1 \neq 0 \), so \( F \setminus \{0, 1\} \) is odd.

We wish to verify that there is at least one vertex which is self invertible. In a contrary way, suppose there is no element in \( F \setminus \{0, 1\} \) which is self invertible. Then take \( x_i \in F \setminus \{0, 1\} \) and its inverse \( x_i^{-1} \) and then pair them \( x_1, x_1^{-1} \). By our supposition for all elements the corresponding inverses exists. If there are \( k \) such pairs formed then \( |F \setminus \{0, 1\}| = 2 \cdot k \) (an even number), a contradiction.

The proof is complete.

References